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Whittaker and Bessel functors for $\mathrm{G}\mathbb{S}\mathrm{p}_4$

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Abstract The theory of Whittaker functors for GL_n is an essential technical tools in Gaitsgory's proof of the Vanishing Conjecture appearing in the geometric Langlands correspondence ([5]). We define Whittaker functors for $\mathrm{G}\mathbb{S}\mathrm{p}_4$ and study their properties. These functors correspond to the maximal parabolic subgroup of $\mathrm{G}\mathbb{S}\mathrm{p}_4$, whose unipotent radical is not commutative.

We also study similar functors corresponding to the Siegel parabolic subgroup of $\mathrm{G}\mathbb{S}\mathrm{p}_4$, they are related with Bessel models for $\mathrm{G}\mathbb{S}\mathrm{p}_4$ and Waldspurger models for GL_2.

We define the Waldspurger category, which is a geometric counterpart of the Waldspurger module over the Hecke algebra of GL_2. We prove a geometric version of the multiplicity one result for the Waldspurger models.

1. Introduction

1.1 Whittaker and Bessel models are of importance in the theory of automorphic representations of $\mathrm{G}\mathbb{S}\mathrm{p}_4$. This paper is the first in a series of two, where we study some phenomena corresponding to these models in the geometric Langlands program.

The theory of Whittaker functors for GL_n is an essential technical tool in Gaitsgory's proof of the Vanishing Conjecture appearing in the geometric Langlands correspondence ([5]). First part of our results is an analog of this theory for $\mathrm{G}\mathbb{S}\mathrm{p}_4$.

Let us first remind some facts about automorphic forms on $G = \mathbb{S}\mathrm{p}_4$. Let X be a smooth projective absolutely irreducible curve over $\mathbb{F}_q$, $F = \mathbb{F}_q(X)$ and $\mathbb{A}$ be the adeles ring of F. Let B be a Borel subgroup of G and $U \subset B$ its unipotent radical. For a character $\psi : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}^*$ one has a global Whittaker module over $G(\mathbb{A})$

$$WM_\psi = \{f : U(F)\backslash G(\mathbb{A}) \to \mathbb{C} \mid \; f(ug) = \psi(u)f(g) \text{ for } u \in U(\mathbb{A}),\; f \text{ is smooth}\}$$

Let $\mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ be the space of cusp forms on $G(F)\backslash G(\mathbb{A})$. The usual Whittaker operator $W_\psi : \mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A})) \to WM_\psi$ is given by

$$W_\psi(f)(g) = \int_{U(F)\backslash U(\mathbb{A})} f(ug)\psi(u^{-1})du,$$

where du is induced from a Haar measure on $U(\mathbb{A})$. Whence for GL_n (and generic ψ) the operator W_ψ is an injection, this is not always the case for more general groups. There are cuspidal automorphic representations of $\mathbb{S}\mathrm{p}_4$ that don't admit a ψ-Whittaker model for any ψ.

Recall that $\mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ decomposes as a direct sum

$$\mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A})) = I_3(H_2) \oplus I_4(H_3) \oplus I_5(H_4) \tag{1}$$

in the notation of ([10], Sect. 1.3, p. 359), the summands being $G(\mathbb{A})$-invariant[1]. The decomposition is orthogonal with respect to the scalar product

$$\langle f, h\rangle = \int_{G(F)\backslash G(\mathbb{A})} f(x)\overline{h(x)}dx, \tag{2}$$

where dx is induced from a Haar measure on $G(\mathbb{A})$.

For any $f \in \mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ its θ-lifting to $\mathbb{O}(2)(\mathbb{A})$ vanishes (*loc.cit.*, Corolary 2 to Theorem I.2.1). By definition, $I_4(H_3) \oplus I_5(H_4)$ (resp., $I_5(H_4)$) are those cuspidal forms whose θ-lifting to $\mathbb{O}_4(\mathbb{A})$ (resp., to $\mathbb{O}_4(\mathbb{A})$ and $\mathbb{O}_6(\mathbb{A})$) vanishes. Here $\mathbb{O}_{2r}$ is the orthogonal group defined by the hyperbolic quadratic form in a $2r$-dimensional space.

The space $I_5(H_4)$ is also the intersection of kernels of W_ψ for all ψ. It is known as the space of hyper-cuspidal forms on $G(F)\backslash G(\mathbb{A})$ ([9], Definition on p. 328). Another description of $I_5(H_4)$ is as follows. Let $P_1 \subset G$ be the parabolic preserving a 1-dimensional isotropic subspace in the standard representation V of G, $U_1 \subset P_1$ be its unipotent radical, U_0 the center of U_1. Then $f \in \mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ lies in $I_5(H_4)$ if and only if

$$\int_{U_0(F)\backslash U_0(\mathbb{A})} f(ug)du = 0$$

for all $g \in G(\mathbb{A})$.

If $V' \subset V$ is a 2-dimensional subspace such that the symplectic form on V restricts to a non degenerate form on V' then let $H \subset G = \mathbb{S}\mathrm{p}(V)$ be the subgroup of those $g \in G$ that preserve and act trivially on V'. Then $f \in \mathcal{A}_{cusp}(G(F)\backslash G(\mathbb{A}))$ lies in $I_4(H_3) \oplus I_5(H_4)$ if and only if

$$\int_{H(F)\backslash H(\mathbb{A})} f(hg)dh = 0$$

for all $g \in G(\mathbb{A})$ (*loc.cit.*, Section 3). Note that $H \widetilde{\rightarrow} \mathrm{SL}_2$.

1.2 In the geometric setting we work with $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$ (over an algebraically closed field of characteristic $p > 2$). For a scheme (or a stack S) write $\mathrm{D}(S)$ for the derived category of ℓ-adic étale sheaves on S.

Let Bun_G be the stack of G-bundles on X. Inside of the triangulated category $\mathrm{D}_{cusp}(\mathrm{Bun}_G)$ of cuspidal sheaves on Bun_G we single out a full triangulated subcategory $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ of hyper-cuspidal sheaves. Both they are preserved by Hecke functors. So, a natural step in the geometric Langlands program for G is to understand the Hecke action on $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ and on $\mathrm{D}_{cusp}(\mathrm{Bun}_G)/\,\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$.

[1] In *loc.cit.* F is a number field, but (1) holds also over function fields.

The category $\mathrm{D}_{cusp}(\mathrm{Bun}_G)$ is equiped with the 'scalar product', which is an analogue of (2), it sends K_1, K_2 to $\mathrm{RHom}(K_1, K_2)$. The (left and right) orthogonal complements ${}^{\perp}\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$, $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)^{\perp} \subset \mathrm{D}_{cusp}(\mathrm{Bun}_G)$ are also preserved by Hecke functors.

A G-bundle on X is a triple: a rank 4 vector bundle M on X, a line bundle $\mathcal{A}$ on X, and a symplectic form $\wedge^2 M \to \mathcal{A}$. Let $\alpha : \bar{\mathcal{Q}}_1 \to \mathrm{Bun}_G$ be the stack over Bun_G whose fibre over $(M, \mathcal{A})$ consists of all nonzero maps of coherent sheaves $\Omega \hookrightarrow M$, where Ω is the canonical line bundle on X.

We introduce the notion of cuspidality and hyper-cuspidality on $\bar{\mathcal{Q}}_1$, thus leading to full triangulated subcategories $\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}(\bar{\mathcal{Q}}_1)$.

Then we describe $\mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$ in terms of geometric Whittaker models. Namely, we introduce a stack $\bar{\mathcal{Q}}$ (it was denoted by $\mathcal{Y}$ in [6]) and a full triangulated subcategory $\mathrm{D}^W(\bar{\mathcal{Q}}) \subset \mathrm{D}(\bar{\mathcal{Q}})$. Our $\mathrm{D}^W(\bar{\mathcal{Q}})$ is a geometric analog of the space WM_ψ.

We define Whittaker functors that give rise to an equivalence of triangulated categories

$$W : \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \widetilde{\to} \mathrm{D}^W(\bar{\mathcal{Q}}) \tag{3}$$

The Hecke functor H^{γ} corresponding to the standard representation of the Langlands dual group $\check{G} \widetilde{\to} \mathrm{G}\mathbb{S}\mathrm{p}_4$ acts on all the categories mentioned in Sect. 1.2. Moreover, the equivalence (3) commutes with H^{γ}. The restriction functor

$$\alpha^* : \mathrm{D}_{cusp}(\mathrm{Bun}_G)/\mathrm{D}_{hcusp}(\mathrm{Bun}_G) \to \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$$

also commutes with H^{γ}. As in the case of GL_n ([5], Theorem 7.9), the advantage of $\bar{\mathcal{Q}}$ over Bun_G is that the functor $\mathrm{H}^{\gamma} : \mathrm{D}(\bar{\mathcal{Q}}) \to \mathrm{D}(X \times \bar{\mathcal{Q}})$ is right-exact for the perverse t-structures.

The essential difference with GL_n case is that the Whittaker functor $W : \mathrm{D}(\bar{\mathcal{Q}}_1) \to \mathrm{D}^W(\bar{\mathcal{Q}})$ is not exact for the perverse t-structures. We can only indicate full triangulated subcategories $\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}(\bar{\mathcal{Q}}_1)$ such that the restriction of W yields an equivalence

$$\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1) \widetilde{\to} \mathrm{D}^W(\bar{\mathcal{Q}})$$

of triangulated categories. Then (3) follows from the fact that the natural inclusion functor induces an equivalence of triangulated categories

$$\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1) \widetilde{\to} \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$$

This is the content of Sect. 2-6.

1.3 The stack $\bar{\mathcal{Q}}_1$ corresponds to the parabolic subgroup $P_1 \subset G$. In Sect. 7 we define functors similar to the Whittaker ones for the Siegel parabolic subgroup $P \subset G$. They are related to Bessel models[2] for G.

The general idea behind is that various Fourier coefficients of automorphic sheaves carry additional structure coming from the action of Hecke operators.

[2] Bessel models will be studied in the second paper of this series.

Let $\alpha_{\mathcal{Z}} : \mathcal{Z}_1 \to \mathrm{Bun}_G$ be the stack whose fibre over $(M, \mathcal{A})$ is the scheme of isotropic subsheaves $L_2 \subset M$, where L_2 is a locally free $\mathcal{O}_X$-module of rank 2. The open substack $\mathrm{Bun}_P \subset \mathcal{Z}_1$ is given by the condition that L_2 is a subbundle. Then Bun_P is the stack classifying: a rank 2 bundle L_2 on X, a line bundle $\mathcal{A}$ on X, and an exact sequence $0 \to \mathrm{Sym}^2 L_2 \to ? \to \mathcal{A} \to 0$.

Let $\mathcal{S}_{ex}$ denote the stack classifying: a rank 2 vector bundle L_2 on X, a line bundle $\mathcal{A}$ on X, and a map $\mathrm{Sym}^2 L_2 \to \mathcal{A} \otimes \Omega$. Write Bun_i for the stack of rank i vector bundles on X. Then Bun_P and $\mathcal{S}_{ex}$ are dual (generalized) vector bundles over $\mathrm{Bun}_2 \times \mathrm{Bun}_1$, so we have the Fourier transform functor $\mathrm{Four} : \mathrm{D}(\mathrm{Bun}_P) \to \mathrm{D}(\mathcal{S}_{ex})$.

For a complex $K \in \mathrm{D}(\mathrm{Bun}_G)$ its Fourier coefficient with respect to the Siegel parabolic is, by definition, $F_{\mathcal{S}_{ex}}(K) = \mathrm{Four}(K \mid_{\mathrm{Bun}_P})$. If K is a Hecke eigen-sheaf on Bun_G then $F_{\mathcal{S}_{ex}}(K)$ satisfies some additional property (cf. Proposition 12), which is a consequence of the following result.

Let $\mathcal{Z}_{2,ex} \to \mathcal{Z}_1$ be the stack whose fibre over a point $(L_2 \subset M, \mathcal{A}) \in \mathcal{Z}_1$ is the space $\mathrm{Hom}(\mathrm{Sym}^2 L_2, \mathcal{A} \otimes \Omega)$. We define a full triangulated subcategory $\mathrm{D}^W(\mathcal{Z}_{2,ex}) \subset \mathrm{D}(\mathcal{Z}_{2,ex})$ singled out by some equivariance condition. Then we establish an equivalence of triangulated categories

$$WZ : \mathrm{D}(\mathcal{Z}_1) \widetilde{\to} \mathrm{D}^W(\mathcal{Z}_{2,ex}),$$

which is exact for perverse t-structures. The Hecke functor H^γ acts on both categories and commutes with this equivalence. Our $\mathrm{D}^W(\mathcal{Z}_{2,ex})$ is a way to think about the Fourier coefficients $F_{\mathcal{S}_{ex}}(K)$ *together with an action of Hecke operators.*

One also has a notion of hyper-cuspidality on $\mathcal{Z}_1$ and $\mathcal{Z}_{2,ex}$ leading to full triangulated subcategories $\mathrm{D}_{hcusp}(\mathcal{Z}_1) \subset \mathrm{D}(\mathcal{Z}_1)$ and $\mathrm{D}^W_{hcusp}(\mathcal{Z}_2) \subset \mathrm{D}^W(\mathcal{Z}_{2,ex})$ preserved by H^γ. The functor WZ induces an equivalence

$$\mathrm{D}_{hcusp}(\mathcal{Z}_1) \widetilde{\to} \mathrm{D}^W_{hcusp}(\mathcal{Z}_2)$$

A complex $K \in \mathrm{D}(\mathrm{Bun}_G)$ is hyper-cuspidal if and only if $\alpha^*_{\mathcal{Z}} K$ is hyper-cuspidal.

1.4 In a sense, Bessel models for G is a way to think about the Fourier coefficients $F_{\mathcal{S}_{ex}}(K)$ of automorphic sheaves $K \in \mathrm{D}(\mathrm{Bun}_G)$ in terms of the Waldspurger models for GL_2 ([3]). This is our motivation for the study of these Walspurger models in Sect. 8, which is independent of the rest of this paper.

The following background result is due to Waldspurger ([12], Lemma 8). Set $F = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]]$. Let $\tilde F$ be an étale F-algebra with $\dim_F(\tilde F) = 2$ such that $\mathbb{F}_q$ is algebraically closed in $\tilde F$. Let $\tilde{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in $\tilde F$. We have two cases:

- $\tilde F \widetilde{\to} \mathbb{F}_q((t^{\frac{1}{2}}))$ (the nonsplit case)
- $\tilde F \widetilde{\to} F \oplus F$ (the split case)

Write $\mathrm{GL}(\tilde F)$ for the automorphism group of the F-vector space $\tilde F$, and $\mathrm{GL}(\tilde{\mathcal{O}}) \subset \mathrm{GL}(\tilde F)$ for the stabilizor of $\tilde{\mathcal{O}}$. Fix a nonramified character $\chi : \tilde F^*/\tilde{\mathcal{O}}^* \to \bar{\mathbb{Q}}_\ell^*$. Denote by $\chi_c : F^*/\mathcal{O}^* \to \bar{\mathbb{Q}}_\ell^*$ the restriction of χ. The *Waldspurger module* is the vector space

$$WA_\chi = \{f : \mathrm{GL}(\tilde F)/\mathrm{GL}(\tilde{\mathcal{O}}) \to \bar{\mathbb{Q}}_\ell \mid f(ux) = \chi(u) f(x) \text{ for } u \in \tilde F^*, \\ f \text{ is of compact support modulo } \tilde F^*\}$$

The Hecke algebra

$$\mathrm{H}_{\chi_c} = \{h : \mathrm{GL}(\tilde{\mathcal{O}})\backslash \mathrm{GL}(\tilde{F})/ \mathrm{GL}(\tilde{\mathcal{O}}) \to \bar{\mathbb{Q}}_\ell \mid h(ux) = \chi_c(u)h(x) \ \text{ for } \ u \in F^*, \\ h \ \text{ is of compact support}\}$$

acts on WA_χ via

$$h \in \mathrm{H}_{\chi_c},\ f \in WA_\chi \ \to \ (h * f)(g) = \int_{\mathrm{GL}(\tilde{F})} h(x)f(gx^{-1})dx,$$

where dx is the Haar measure of $\mathrm{GL}(\tilde{F})$ such that the volume of $\mathrm{GL}(\tilde{\mathcal{O}})$ is one. Then WA_χ is *a free module of rank one* over H_{χ_c} (mutilpicity one for Waldspurger model).

We prove a categorical version of this. Namely, the affine grassmanian $\mathrm{Gr}_{\tilde{F}} := \mathrm{GL}(\tilde{F})/ \mathrm{GL}(\tilde{\mathcal{O}})$ can be viewed as an ind-scheme over $\mathbb{F}_q$ equiped with an action of the group scheme $\tilde{F}^*$. Pick a 1-dimensional $\bar{\mathbb{Q}}_\ell$-vector space $\tilde{E}_{\tilde{x}}$ for each $\tilde{x} \in \operatorname{Spec} \tilde{F}$. We introduce *Waldspurger category* of those $\tilde{\mathcal{O}}^*$-equivariant perverse sheaves on $\mathrm{Gr}_{\tilde{F}}$ that change under the action of each uniformizor $t_{\tilde{x}} \in \tilde{F}^*/\tilde{\mathcal{O}}^*$ by $\tilde{E}_{\tilde{x}}$ (for each $\tilde{x} \in \operatorname{Spec} \tilde{F}$). This is a geometric counterpart of WA_χ.

The nonramified Hecke algebra for GL_2 also admits a geometric counetrpart, the category $\mathrm{Sph}(\mathrm{Gr}_{\tilde{F}})$ of $\mathrm{GL}_2(\tilde{\mathcal{O}})$-equivariant perverse sheaves on $\mathrm{Gr}_{\tilde{F}}$. This is a tensor category equivalent to the category of representations of GL_2 ([7]). It acts on the Waldspurger category by convolutions.

Actually we work with a global version $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}^x_\pi)$ of the Waldspurger category (in geometric setting we replace $\mathbb{F}_q$ by an algebraically closed field k of characteristic $p > 2$). The input data for our definition of $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}^x_\pi)$ is a two-sheeted covering $\pi : \tilde{X} \to X$ ramified at some divisor D_π on X, a point $x \in X$, and a rank one local system $\tilde{E}$ on $\tilde{X}$. Here $\tilde{X}$ and X are smooth projective curves over k (with X connected).

Objects of $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}^x_\pi)$ are some perverse sheaves on a stack $\mathcal{W}\mathrm{ald}^x_\pi$, which is a global model of 'the space' of $\tilde{F}^*$-orbits on $\mathrm{Gr}_{\tilde{F}}$. By definition, $\mathcal{W}\mathrm{ald}^x_\pi$ classifies collections: a rank 2 vector bundle L on X, a line bundle $\mathcal{B}$ on $\pi^{-1}(X - x)$, and an isomorphism $\pi_*\mathcal{B} \widetilde{\to} L \mid_{X-x}$.

Our main result here is Theorem 4 describing the action of $\mathrm{Sph}(\mathrm{Gr}_{\tilde{F}})$ on irreducible objects of $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}^x_\pi)$. It implies the above cited multiplicity one for the Waldspurger models. This circle of ideas is very much inspired by [4]. Note that, to the difference with the case of Whittaker categories studied in *loc.cit.*, the category $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}^x_\pi)$ is not semi-simple.

2. Whittaker categories

2.1 Notation Let k denote an algebraically closed field of characteristic $p > 2$. All the schemes (or stacks) we consider are defined over k. Let X be a smooth projective connected curve. Fix a prime $\ell \neq p$. For a scheme (or stack) S write $\mathrm{D}(S)$ for the bounded derived category of ℓ-adic étale sheaves on S, and $\mathrm{P}(S) \subset \mathrm{D}(S)$ for the category of perverse sheaves.

Fix a nontrivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathcal{L}_\psi$ the corresponding Artin-Shreier sheaf on $\mathbb{A}^1$. The Fourier transform functors will be always normalized to preserve perversity and purity.

Let $G = \mathbb{GSp}_4$, so G is the quotient of $\mathbb{G}_m \times \mathbb{Sp}_4$ by the diagonally embedded $\{\pm 1\}$. Denote by $\check{G}$ the Langlands dual group to G (over $\bar{\mathbb{Q}}_\ell$). We use the following notation from ([6], example 2 in the appendix). The group G is realized as the subgroup of $\mathrm{GL}(k^4)$ preserving up to a scalar the bilinear form given by the matrix

$$\left(\begin{array}{cc} 0 & E_2 \\ -E_2 & 0 \end{array}\right),$$

where E_2 is the unit matrix of GL_2.

Let T be the maximal torus of G given by $\{(y_1, \dots, y_4) \mid y_i y_{2+i}$ does not depend on $i\}$. Let Λ (resp., $\check{\Lambda}$) denote the coweight (resp., weight) lattice of T. Write $V^{\check{\lambda}}$ for the irreducible representation of G of highest weight $\check{\lambda}$.

Let $\check{\epsilon}_i \in \check{\Lambda}$ be the character that sends a point of T to y_i. We have $\Lambda = \{(a_1, \dots, a_4) \in \mathbb{Z}^4 \mid a_i + a_{2+i}$ does not depend on $i\}$ and

$$\check{\Lambda} = \mathbb{Z}^4 / \{\check{\epsilon}_1 + \check{\epsilon}_3 - \check{\epsilon}_2 - \check{\epsilon}_4\}$$

Let $P_1 \subset G$ be the parabolic subgroup preserving the isotropic subspace ke_1. Let $P_2 \subset G$ denote the Borel subgroup preserving the flag $ke_1 \subset ke_1 \oplus ke_2$ of isotropic subspaces. Here $\{e_i\}$ is the standard basis of k^4. Let U_i be the unipotent radical of P_i and $M_1 = P_1/U_1$.

The simple roots are $\check{\alpha}_1 = \check{e}_1 - \check{e}_2$ and $\check{\alpha}_2 = \check{e}_2 - \check{e}_4$. The half sum of positive roots of G is denoted by $\check{\rho} \in \check{\Lambda}$.

Let $P \subset G$ denote the Siegel parabolic preserving the lagrangian subspace $ke_1 \oplus ke_2 \subset k^4$. Let $U \subset P$ be its unipotent radical and $M = P/U$.

Set $\gamma = (1, 1; 0, 0) \in \Lambda$, this is the dominant coweight corresponding to the standard representation of $\check{G} \widetilde{\to} \mathbb{GSp}_4$. Fix fundamental weights $\check{\omega}_1 = (1, 0, 0, 0)$ and $\check{\omega}_2 = (1, 1, 0, 0)$. So, $V^{\check{\omega}_1}$ is the standard representation. The orthogonal to the coroot lattice is $\mathbb{Z}\check{\omega}_0$ with $\check{\omega}_0 = (1, 0, 1, 0)$.

Note that the symplectic form $\wedge^2 V^{\check{\omega}_1} \to V^{\check{\omega}_0}$ induces an isomorphism $\det V^{\check{\omega}_1} \widetilde{\to} (V^{\check{\omega}_0})^{\otimes 2}$.

2.2 Hecke functor Let Bun_G denote the stack of G-bundles on X. For a G-bundle $\mathcal{F}_G$ let $M = V^{\check{\omega}_1}_{\mathcal{F}_G}$, $\mathcal{W} = V^{\check{\omega}_2}_{\mathcal{F}_G}$ and $\mathcal{A} = V^{\check{\omega}_0}_{\mathcal{F}_G}$. In this way Bun_G becomes the stack classifying the data: a line bundle $\mathcal{A}$ on X, a vector bundle M of rank 4 on X with a symplectic form $\wedge^2 M \to \mathcal{A}$. The exact sequence

$$0 \to \mathcal{W} \to \wedge^2 M \to \mathcal{A} \to 0$$

splits canonically.

Denote by $\mathcal{H}_G$ the stack of collections: $x \in X, \mathcal{F}_G, \mathcal{F}'_G \in \mathrm{Bun}_G$ and $\mathcal{F}_G \widetilde{\to} \mathcal{F}'_G \mid_{X-x}$ such that $\mathcal{F}_G$ is in the position γ with respect to $\mathcal{F}'_G$. In other words, we have $\mathcal{A}' = \mathcal{A}(x)$, $M \subset M'$, the diagrams commute

$$\begin{array}{ccc} \wedge^2 M' & \to & \mathcal{A}' \\ \uparrow & & \uparrow \\ \wedge^2 M & \to & \mathcal{A} \end{array}$$

and

$$\begin{array}{ccc} \det M' & \widetilde{\to} & \mathcal{A}'^2 \\ \uparrow & & \uparrow \\ \det M & \widetilde{\to} & \mathcal{A}^2 \end{array}$$

and $M/M'(-x) \subset M'/M'(-x)$ is a lagrangian subspace.

We have a diagram $\mathrm{Bun}_G \stackrel{\mathfrak{p}}{\leftarrow} \mathcal{H}_G \stackrel{\mathfrak{q}}{\to} \mathrm{Bun}_G$, where the map $\mathfrak{p}$ (resp., $\mathfrak{q}$) sends the above collection to $\mathcal{F}_G$ (resp., $\mathcal{F}'_G$). Let $\mathrm{supp} : \mathcal{H}_G \to X$ be the map sending the above point to x. Note that $\mathfrak{q}$ is smooth of relative dimension $1 + \langle \gamma, 2\check{\rho} \rangle$. Let

$$\mathrm{H} : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_G)$$

denote the Hecke functor corresponding to γ, that is,

$$\mathrm{H}(K) = (\mathrm{supp} \times \mathfrak{p})_! \mathfrak{q}^* K \otimes \bar{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes 1 + \langle \gamma, 2\check{\rho} \rangle}$$

2.3 Drinfeld compactifications We fix a particular T-torsor on X with trivial conductor $(\mathcal{F}_T, \tilde{\omega})$ by requiring $\mathcal{L}^{\check{\omega}_1}_{\mathcal{F}_T} \widetilde{\to} \Omega$. The pair $(\mathcal{F}_T, \tilde{\omega})$ with this property is defined up to a unique isomorphism, and we have $\mathcal{L}^{\check{\omega}_2}_{\mathcal{F}_T} \widetilde{\to} \Omega$ and $\mathcal{L}^{\check{\omega}_0}_{\mathcal{F}_T} \widetilde{\to} \Omega^{-1}$.

For $k = 1, 2, 3$ define the stack $\bar{\mathcal{Q}}_k$ as follows. It classifies a point $\mathcal{F}_G \in \mathrm{Bun}_G$ together with sections $t_1, \ldots, t_k$ satisfying Plucker relations, where

$$\begin{array}{l} t_1 : \Omega \hookrightarrow M \\ t_2 : \Omega \hookrightarrow \mathcal{W} \\ t_3 : \Omega^{-1} \hookrightarrow \mathcal{A} \end{array}$$

It is understood that Plucker relations are empty for $k = 1$, and for $k = 2, 3$ they mean that, at the generic point of X, the sections $t_1, \ldots, t_k$ come from a B-structure on $\mathcal{F}_G$.

Set $\bar{\mathcal{Q}} = \bar{\mathcal{Q}}_3$. Let also $\bar{\mathcal{Q}}_{k,ex}$ be the stack defined in the same way as $\bar{\mathcal{Q}}_k$ with the only difference that the last section t_k is not necessairy an inclusion (here 'ex' stands for 'extended'). So, $\bar{\mathcal{Q}}_k \subset \bar{\mathcal{Q}}_{k,ex}$ is an open substack.

Denote by $\pi_{k+1,k} : \bar{\mathcal{Q}}_{k+1} \to \bar{\mathcal{Q}}_k$ and $\pi_{k+1,k,ex} : \bar{\mathcal{Q}}_{k+1,ex} \to \bar{\mathcal{Q}}_k$ the natural forgetful maps.

For each k we have the diagram

$$\bar{\mathcal{Q}}_{k,ex} \stackrel{\mathfrak{p}_{k,ex}}{\leftarrow} \bar{\mathcal{Q}}_{k,ex} \times_{\mathrm{Bun}_G} \mathcal{H}_G \stackrel{\mathfrak{q}_{k,ex}}{\to} \bar{\mathcal{Q}}_{k,ex},$$

where we used the map $\mathfrak{p} : \mathcal{H}_G \to \mathrm{Bun}_G$ in the definition of the fibred product, $\mathfrak{p}_{k,ex}$ is the projection, and $\mathfrak{q}_{k,ex}$ sends a point of $\bar{\mathcal{Q}}_{k,ex} \times_{\mathrm{Bun}_G} \mathcal{H}_G$ to $(\mathcal{F}'_G, t'_1, \ldots, t'_k)$ with t'_i being the compositions

$$\begin{array}{l} t_1 : \Omega \to M \hookrightarrow M' \\ t_2 : \Omega \to \mathcal{W} \hookrightarrow \mathcal{W}' \\ t_3 : \Omega^{-1} \to \mathcal{A} \hookrightarrow \mathcal{A}' \end{array}$$

For $k=1,2,3$ we have the functor $\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}} : \mathrm{D}(\bar{\mathcal{Q}}_{k,ex}) \to \mathrm{D}(X \times \bar{\mathcal{Q}}_{k,ex})$ given by

$$\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}(K) = (\mathrm{supp} \times \mathfrak{p}_{k,ex})_! \mathfrak{q}^*_{k,ex} K \otimes \bar{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes \langle \gamma, 2\check{\rho} \rangle}$$

The restriction of $\mathfrak{q}_{k,ex}$ to $\bar{\mathcal{Q}}_k \times_{\mathrm{Bun}_G} \mathcal{H}_G$ factors through $\bar{\mathcal{Q}}_k \subset \bar{\mathcal{Q}}_{k,ex}$. So, we also have diagrams

$$\bar{\mathcal{Q}}_k \stackrel{\mathfrak{p}_k}{\leftarrow} \bar{\mathcal{Q}}_k \times_{\mathrm{Bun}_G} \mathcal{H}_G \stackrel{\mathfrak{q}_k}{\to} \bar{\mathcal{Q}}_k,$$

where $\mathfrak{p}_k$ (resp., $\mathfrak{q}_k$) is the restriction of $\mathfrak{p}_{k,ex}$ (resp., of $\mathfrak{q}_{k,ex}$). For $k=1,2,3$ denote by

$$\mathrm{H}^{\bar{\mathcal{Q}}_k} : \mathrm{D}(\bar{\mathcal{Q}}_k) \to \mathrm{D}(X \times \bar{\mathcal{Q}}_k)$$

the functor given by

$$\mathrm{H}^{\bar{\mathcal{Q}}_k}(K) = (\mathrm{supp} \times \mathfrak{p}_k)_! \mathfrak{q}^*_k K \otimes \bar{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes \langle \gamma, 2\check{\rho} \rangle} \tag{4}$$

The projection $\alpha : \bar{\mathcal{Q}}_1 \to \mathrm{Bun}_G$ fits into the diagram

$$\begin{array}{ccccc}
\bar{\mathcal{Q}}_1 & \stackrel{\mathfrak{p}_1}{\leftarrow} & \bar{\mathcal{Q}}_1 \times_{\mathrm{Bun}_G} \mathcal{H}_G & \stackrel{\mathfrak{q}_1}{\to} & \bar{\mathcal{Q}}_1 \\
\downarrow \alpha & & \downarrow & & \downarrow \alpha \\
\mathrm{Bun}_G & \stackrel{\mathfrak{p}}{\leftarrow} & \mathcal{H}_G & \stackrel{\mathfrak{q}}{\to} & \mathrm{Bun}_G,
\end{array}$$

in which the left square is cartesian. So, $(\mathrm{id} \times \alpha)^* \circ \mathrm{H} \widetilde{\to} \mathrm{H}^{\bar{\mathcal{Q}}_1} \circ \alpha^*[1](\frac{1}{2})$ naturally. Over the open substack of Bun_G given by $\mathrm{Ext}^1(\Omega, M) = 0$, the map $\alpha : \bar{\mathcal{Q}}_1 \to \mathrm{Bun}_G$ is smooth.

2.4 Let $\pi_{0,1,ex} : \bar{\mathcal{Q}}_{0,ex} \to \bar{\mathcal{Q}}_1$ be the vector bundle with fibre consisting of all sections $t_0 : \Omega \to \mathcal{A}$. Let $i_0 : \bar{\mathcal{Q}}_1 \to \bar{\mathcal{Q}}_{0,ex}$ denote the zero section and $j : \bar{\mathcal{Q}}_0 \subset \bar{\mathcal{Q}}_{0,ex}$ its complement given by: t_0 is an inclusion.

We have the diagram

$$\bar{\mathcal{Q}}_{0,ex} \stackrel{\mathfrak{p}_{0,ex}}{\to} \bar{\mathcal{Q}}_{0,ex} \times_{\mathrm{Bun}_G} \mathcal{H}_G \stackrel{\mathfrak{q}_{0,ex}}{\to} \bar{\mathcal{Q}}_{0,ex},$$

where we used $\mathfrak{p} : \mathcal{H}_G \to \mathrm{Bun}_G$ in the definition of the fibred product, $\mathfrak{p}_{0,ex}$ is the projection, and $\mathfrak{q}_{0,ex}$ sends a point of $\bar{\mathcal{Q}}_{0,ex} \times_{\mathrm{Bun}_G} \mathcal{H}_G$ to $(\mathcal{F}'_G, t'_0, t'_1)$. Here, as above, t'_i are the compositions

$$\begin{array}{c} t_0 : \Omega \to \mathcal{A} \hookrightarrow \mathcal{A}' \\ t_1 : \Omega \hookrightarrow M \hookrightarrow M' \end{array}$$

Restricting, one gets the diagram $\bar{\mathcal{Q}}_0 \stackrel{\mathfrak{p}_0}{\to} \bar{\mathcal{Q}}_0 \times_{\mathrm{Bun}_G} \mathcal{H}_G \stackrel{\mathfrak{q}_0}{\to} \bar{\mathcal{Q}}_0,$. The functors

$$\mathrm{H}^{\bar{\mathcal{Q}}_{0,ex}} : \mathrm{D}(\bar{\mathcal{Q}}_{0,ex}) \to \mathrm{D}(X \times \bar{\mathcal{Q}}_{0,ex})$$

and $\mathrm{H}^{\bar{\mathcal{Q}}_0} : \mathrm{D}(\bar{\mathcal{Q}}_0) \to \mathrm{D}(X \times \bar{\mathcal{Q}}_0)$ are defined as in (4).

Remark 1. For any $K \in \mathrm{D}(\bar{\mathcal{Q}}_{0,ex})$ we have a natural isomorphism of distinguished triangles

$$\begin{array}{ccccc} j_! \mathrm{H}^{\bar{\mathcal{Q}}_0}(j^* K) & \to & \mathrm{H}^{\bar{\mathcal{Q}}_{0,ex}}(K) & \to & (i_0)_* \mathrm{H}^{\bar{\mathcal{Q}}_1}(i_0^* K) \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow \\ j_! j^* \mathrm{H}^{\bar{\mathcal{Q}}_{0,ex}}(K) & \to & \mathrm{H}^{\bar{\mathcal{Q}}_{0,ex}}(K) & \to & (i_0)_* i_0^* \mathrm{H}^{\bar{\mathcal{Q}}_{0,ex}}(K) \end{array}$$

2.5 Categories to construct We will introduce triangulated categories $\mathrm{D}^W(\bar{\mathcal{Q}}_k)$ (resp., $\mathrm{D}^W(\bar{\mathcal{Q}}_{k,ex})$) of sheaves on $\bar{\mathcal{Q}}_k$ (resp., on $\bar{\mathcal{Q}}_{k,ex}$) for $k = 0, 1, 2, 3$ (resp, for $k = 0, 2, 3$).

Each $\mathrm{D}^W(\bar{\mathcal{Q}}_k)$ will be a full triangulated subcategory of $\mathrm{D}(\bar{\mathcal{Q}}_k)$ defined by the condition that $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_k)$ if its perverse cohomology belong to a certain Serre subcategory $\mathrm{P}^W(\bar{\mathcal{Q}}_k)$ singled out by some equivariance condition; and similarly for $\mathrm{D}^W(\bar{\mathcal{Q}}_{k,ex})$.

Though we don't reflect this in the notation, all our equivariant categories (except $\mathrm{D}^W(\bar{\mathcal{Q}}_1)$) will depend on the character ψ.

2.6 Let $y \in X$ be a closed point. For $k = 1, 2$ let $\bar{\mathcal{Q}}_k^y \subset \bar{\mathcal{Q}}_k$ be the open substack given by the condition that neither of the maps $t_1, \dots, t_k$ has zero at y.

If $(\mathcal{F}_G, t_1, \dots, t_k)$ is a point of $\bar{\mathcal{Q}}_k^y$ then over the formal disk D_y at y we obtain a P_k-torsor $\mathcal{F}_{P_k}$. Let $N_{k,y} \to \bar{\mathcal{Q}}_k^y$ be stack whose fibre over a point of $\bar{\mathcal{Q}}_k^y$ is

$$\mathrm{H}^0(D_y^*,\ \mathcal{F}_{P_k} \times_{P_k} U_k)$$

This is an ind-groupscheme over $\bar{\mathcal{Q}}_k^y$, it can be represented as a union of group schemes ${}^i N_{k,y}$ for $i \in \mathbb{N}$, where ${}^i N_{k,y} \hookrightarrow {}^{i+1} N_{k,y}$ is a closed immersion, and ${}^i N_{k,y}/{}^0 N_{k,y}$ is of finite type over $\bar{\mathcal{Q}}_k^y$ for $i > 0$. We assume that the fibre of ${}^0 N_{k,y} \to \bar{\mathcal{Q}}_k^y$ is

$$\mathrm{H}^0(D_y,\ \mathcal{F}_{P_k} \times_{P_k} U_k)$$

Let $\mathcal{H}_{k,y} \to \bar{\mathcal{Q}}_k^y$ denote the stack over $\bar{\mathcal{Q}}_k^y$ with fibre $N_{k,y}/{}^0 N_{k,y}$. This is an ind-scheme over $\bar{\mathcal{Q}}_k^y$, and we have

$$\mathcal{H}_{k,y} = \bigcup_i {}^i \mathcal{H}_{k,y},$$

where ${}^i \mathrm{pr}_k : {}^i \mathcal{H}_{k,y} \to \bar{\mathcal{Q}}_k^y$ is the stack with fibre ${}^i N_{k,y}/{}^0 N_{k,y}$.

2.7 Groupoids As in ([5], sect. 4.3) one endows $\mathcal{H}_{k,y}$ with the structure of a groupoid over $\bar{\mathcal{Q}}_k^y$. We denote by

$${}^i \mathrm{act}_k : {}^i \mathcal{H}_{k,y} \to \bar{\mathcal{Q}}_k^y$$

the restriction of the action map.

For $k = 1, 2$ define the open substack $\bar{\mathcal{Q}}_{k+1,ex}^y \subset \bar{\mathcal{Q}}_{k+1,ex}$ as $\bar{\mathcal{Q}}_{k+1,ex}^y = \bar{\mathcal{Q}}_k^y \times_{\bar{\mathcal{Q}}_k} \bar{\mathcal{Q}}_{k+1,ex}$. The groupoid $\mathcal{H}_{k,y} \to \bar{\mathcal{Q}}_k^y$ "lifts" to $\bar{\mathcal{Q}}_{k+1,ex}^y$. In other words,

$$\mathcal{H}_{k,y} \times_{\bar{\mathcal{Q}}_k^y} \bar{\mathcal{Q}}_{k+1,ex}^y$$

has a structure of a groupoid over $\bar{\mathcal{Q}}^y_{k+1,ex}$ (we used the projections to define the above fibre product). Moreover, the diagram is cartesian

$$\begin{array}{ccc} \mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex} & \stackrel{\mathrm{act}}{\to} & \bar{\mathcal{Q}}^y_{k+1,ex} \\ \downarrow \mathrm{id}\times\pi_{k+1,k,ex} & & \downarrow \pi_{k+1,k,ex} \\ \mathcal{H}_{k,y} & \stackrel{\mathrm{act}}{\to} & \bar{\mathcal{Q}}^y_k \end{array}$$

Denote by

$$^i\operatorname{act}_{k,ex}: {}^i\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex}\to\bar{\mathcal{Q}}^y_{k+1,ex}$$

the action map.

Let $\bar{\mathcal{Q}}^y_{0,ex}\subset\bar{\mathcal{Q}}_{0,ex}$ be the preimage of $\bar{\mathcal{Q}}^y_1$ under $\pi_{0,1,ex}:\bar{\mathcal{Q}}_{0,ex}\to\bar{\mathcal{Q}}_1$. The groupoid $\mathcal{H}_{1,y}\to\bar{\mathcal{Q}}^y_1$ "lifts" to $\bar{\mathcal{Q}}^y_{0,ex}$ in the same sense as above.

2.8 We single out the subgroupoid $\mathcal{H}_{0,y}\subset\mathcal{H}_{1,y}$ as follows.

Let $U_0\subset U_1$ denote the center of U_1. The exact sequence $1\to U_0\to U_1\to U_1/U_0\to 1$ does not split, we have $U_0\,\widetilde{\to}\,\mathbb{G}_a$ and $U_1/U_0\,\widetilde{\to}\,\mathbb{G}_a^2$.

The stack Bun_{P_1} classifies: a G-torsor $\mathcal{F}_G$ together with a line subbundle $L_1\subset M$. Note that L_1 is automatically isotropic and denote by $L_{-1}\subset M$ its orthogonal complement. For such $\mathcal{F}_{P_1}\in\mathrm{Bun}_{P_1}$ the vector bundle $\mathcal{F}_{P_1}\times_{P_1}U_0$ is $L_1^2\otimes\mathcal{A}^{-1}$. It is understood that P_1 acts on U_0 adjointly.

By definition, the fibre of $\mathcal{H}_{0,y}\to\bar{\mathcal{Q}}^y_1$ is

$$\mathrm{H}^0(D^*_y,\,\mathcal{F}_{P_1}\times_{P_1}U_0)/\mathrm{H}^0(D_y,\,\mathcal{F}_{P_1}\times_{P_1}U_0)\,\widetilde{\to}\,\Omega^2\otimes\mathcal{A}^{-1}(\infty y)/\Omega^2\otimes\mathcal{A}^{-1}$$

Denote by ${}^i\mathcal{H}_{0,y}\subset\mathcal{H}_{0,y}$ the subgroupoid with fibre

$$\Omega^2\otimes\mathcal{A}^{-1}(iy)/\Omega^2\otimes\mathcal{A}^{-1}$$

We write ${}^i\operatorname{pr}_0: {}^i\mathcal{H}_{0,y}\to\bar{\mathcal{Q}}^y_1$ for the projection and ${}^i\operatorname{act}_0: {}^i\mathcal{H}_{0,y}\to\bar{\mathcal{Q}}^y_1$ for the action map.

Let also

$$^i\operatorname{act}_{0,ex}: {}^i\mathcal{H}_{0,y}\times_{\bar{\mathcal{Q}}^y_1}\bar{\mathcal{Q}}^y_{0,ex}\to\bar{\mathcal{Q}}^y_{0,ex}$$

denote the action map.

2.9 Characters Let us construct a natural map

$$\chi_{0,y}:\mathcal{H}_{0,y}\times_{\bar{\mathcal{Q}}^y_1}\bar{\mathcal{Q}}^y_{0,ex}\to\mathbb{A}^1$$

The element $t_0:\Omega\to\mathcal{A}$ gives rise to a morphism

$$\Omega^2\otimes\mathcal{A}^{-1}(iy)/\Omega^2\otimes\mathcal{A}^{-1}\to\Omega(\infty y)/\Omega$$

and we take the residue of the image of $g\in\mathcal{H}_{0,y}$ under this map.

Let us construct for $k=1,2$ a natural map

$$\chi_{k,y}:\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\ \bar{\mathcal{Q}}^y_{k+1,ex}\to\mathbb{A}^1$$

CASE $k=1$**.**
If $\mathcal{F}_{P_1}$ is a P_1-torsor on a scheme given by $(L_1\subset L_{-1}\subset M)$ then the vector bundle $\mathcal{F}_{P_1}\times_{P_1}U_1/U_0$ is $\mathrm{Hom}(L_{-1}/L_1,\ L_1)$.

Recall that a point of $\bar{\mathcal{Q}}^y_1$ defines a P_1-torsor $\mathcal{F}_{P_1}$ on D_y. Let $\mathcal{E}_1\to\bar{\mathcal{Q}}^y_1$ be the stack whose fibre over a point of $\bar{\mathcal{Q}}^y_1$ is

$$\mathrm{H}^0(D^*_y,\ \mathcal{F}_{P_1}\times_{P_1}U_1/U_0)/\mathrm{H}^0(D_y,\ \mathcal{F}_{P_1}\times_{P_1}U_1/U_0)\widetilde{\to}(L_{-1}/\Omega)^*\otimes(\Omega(\infty y)/\Omega)$$

We have a natural map $\mathcal{H}_{1,y}\to\mathcal{E}_1$ over $\bar{\mathcal{Q}}^y_1$. Given a point of $\bar{\mathcal{Q}}^y_{2,ex}$, over D_y the section $t_2:\Omega\to\mathcal{W}$ gives rise to a map $s:\mathcal{O}\to L_{-1}/\Omega$ such that $t_2=t_1\wedge s$. By definition, $\chi_{1,y}$ is the residue of the pairing of s with the image of $g\in\mathcal{H}_{1,y}$ in $\mathcal{E}_1$.

CASE $k=2$**.**
Given a point of $\bar{\mathcal{Q}}^y_2$ we obtain a P_2-torsor $\mathcal{F}_{P_2}$ over D_y. Let $U_{2,ab}$ be the abelinization of U_2 then

$$\mathrm{H}^0(D^*_y,\ \mathcal{F}_{P_2}\times_{P_2}U_{2,ab})/\mathrm{H}^0(D_y,\ \mathcal{F}_{P_2}\times_{P_2}U_{2,ab})\widetilde{\to}\Omega(\infty y)/\Omega\oplus\mathcal{A}^{-1}(\infty y)/\mathcal{A}^{-1},\qquad(5)$$

where the two summands correspond to the simple roots of G. To define $\chi_{2,y}$, we take the image of $g\in\mathcal{H}_{2,y}$ in (5), pair it with $t_3:\Omega^{-1}\to\mathcal{A}$ and take the sum of residues.

For $k=1,2$ write

$$^i\chi_{k,y}:{}^i\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\ \bar{\mathcal{Q}}^y_{k+1,ex}\to\mathbb{A}^1$$

for the restriction of $\chi_{k,y}$, and similarly for $^i\chi_{0,y}$.

2.10 Categories on $\bar{\mathcal{Q}}^y_{k+1,ex}$

For $k=1,2$ define the full subcategory $\mathrm{P}^W(\bar{\mathcal{Q}}^y_{k+1,ex})\subset\mathrm{P}(\bar{\mathcal{Q}}^y_{k+1,ex})$ to consist of all perverse sheaves $K\in\mathrm{P}(\bar{\mathcal{Q}}^y_{k+1,ex})$ with the property:

For any $i\in\mathbb{N}$ there is an isomorphism on $^i\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\ \bar{\mathcal{Q}}^y_{k+1,ex}$

$$^i\chi^*_{k,y}(\mathcal{L}_\psi)\otimes\mathrm{pr}_2^*K\widetilde{\to}{}^i\,\mathrm{act}^*_{k,ex}K$$

whose restriction to the unit section $\bar{\mathcal{Q}}^y_{k+1,ex}\subset{}^i\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\ \bar{\mathcal{Q}}^y_{k+1,ex}$ is the identity map.

Similarly, $\mathrm{P}^W(\bar{\mathcal{Q}}^y_{0,ex})\subset\mathrm{P}(\bar{\mathcal{Q}}^y_{0,ex})$ is the full subcategory consisiting of perverse sheaves K with the property:

For any $i\in\mathbb{N}$ there is an isomorphism on $^i\mathcal{H}_{0,y}\times_{\bar{\mathcal{Q}}^y_1}\bar{\mathcal{Q}}^y_{0,ex}$

$$^i\chi^*_{0,y}(\mathcal{L}_\psi)\otimes\mathrm{pr}_2^*K\widetilde{\to}{}^i\,\mathrm{act}^*_{0,ex}K$$

whose restriction to the unit section $\bar{\mathcal{Q}}^y_{0,ex} \subset {}^i\mathcal{H}_{0,y} \times_{\bar{\mathcal{Q}}^y_1} \bar{\mathcal{Q}}^y_{0,ex}$ is the identity map.

For $k = -1, 1, 2$ as in ([5], Sect. 4.7-4.8) one shows that $\mathrm{P}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$ is a Serre subcategory of $\mathrm{P}(\bar{\mathcal{Q}}^y_{k+1,ex})$. Then $\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex}) \subset \mathrm{D}(\bar{\mathcal{Q}}^y_{k+1,ex})$ is a full triangulated subcategory consisting of objects whose perverse cohomology belong to $\mathrm{P}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$.

2.11 In all the three cases $k = -1, 1, 2$ define $\mathrm{P}^W(\bar{\mathcal{Q}}_{k+1,ex}) \subset \mathrm{P}(\bar{\mathcal{Q}}_{k+1,ex})$ as the full subcategory consisting of $K \in \mathrm{P}(\bar{\mathcal{Q}}_{k+1,ex})$ such that

$$K \mid_{\bar{\mathcal{Q}}^y_{k+1,ex}} \in \mathrm{P}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$$

for any $y \in X$. Then $\mathrm{P}^W(\bar{\mathcal{Q}}_{k+1,ex})$ is a Serre subcategory of $\mathrm{P}(\bar{\mathcal{Q}}_{k+1,ex})$. Set $\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ to be the full triangulated subcategory of $\mathrm{D}(\bar{\mathcal{Q}}_{k+1,ex})$ generated by $\mathrm{P}^W(\bar{\mathcal{Q}}_{k+1,ex})$.

For $k = -1, 1, 2$ we also have the categories $\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1})$ and $\mathrm{P}^W(\bar{\mathcal{Q}}_{k+1})$ defined in a similar fashion, because the open substack $\bar{\mathcal{Q}}_{k+1} \subset \bar{\mathcal{Q}}_{k+1,ex}$ is preserved by the action of the corresponding groupoid.

2.12 Recall the vector bundle $\pi_{0,1,ex} : \bar{\mathcal{Q}}_{0,ex} \to \bar{\mathcal{Q}}_1$. Let $i_0 : \bar{\mathcal{Q}}_1 \hookrightarrow \bar{\mathcal{Q}}_{0,ex}$ denote its zero section and $j : \bar{\mathcal{Q}}_0 \hookrightarrow \bar{\mathcal{Q}}_{0,ex}$ the complement to the zero section.

Let $\mathrm{D}^W(\bar{\mathcal{Q}}_1) \subset \mathrm{D}(\bar{\mathcal{Q}}_1)$ be the full triangulated subcategory consisiting of those $K \in \mathrm{D}(\bar{\mathcal{Q}}_1)$ for which $(i_0)_! K \in \mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex})$. The Serre subcategory $\mathrm{P}^W(\bar{\mathcal{Q}}_1) \subset \mathrm{P}(\bar{\mathcal{Q}}_1)$ is defined by the same condition.

In other words, $K \in \mathrm{P}(\bar{\mathcal{Q}}_1)$ lies in $\mathrm{P}^W(\bar{\mathcal{Q}}_1)$ if and only if it is invariant under the action of the groupoids $\mathcal{H}_{0,y}$ for all $y \in X$.

Note that any $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex})$ fits into a distinguished triangle $j_! j^* K \to K \to (i_0)_! i_0^* K$ with $i_0^* K \in \mathrm{D}^W(\bar{\mathcal{Q}}_1)$ and $j^* K \in \mathrm{D}^W(\bar{\mathcal{Q}}_0)$.

2.13 Stratifications

For $k = 1, 2$ stratify $\bar{\mathcal{Q}}_k$ as follows. For a string of nonnegative integers $\bar{d} = (d_1, \ldots, d_k)$ let ${}^{\bar{d}}\bar{\mathcal{Q}}_k \subset \bar{\mathcal{Q}}_k$ be the locally closed substack given by: there exist $D_1 \in X^{(d_1)}, \ldots, D_k \in X^{(d_k)}$ such that

$$t_i : \mathcal{L}^{\check{\omega}_i}_{\mathcal{F}_T}(D_i) \hookrightarrow V^{\check{\omega}_i}_{\mathcal{F}_G}$$

is a subbundle for $i = 1, \ldots, k$. In other words, $\Omega(D_1) \subset M$ is a subbundle for $k = 1$; and for $k = 2$ there is one more condition: $\Omega(D_2) \subset \mathcal{W}$ is a subbundle.

Let ${}^{\bar{d}}\bar{\mathcal{Q}}^y_k$ be the preimage of $\bar{\mathcal{Q}}^y_k$ under ${}^{\bar{d}}\bar{\mathcal{Q}}_k \hookrightarrow \bar{\mathcal{Q}}_k$. The stack ${}^{\bar{d}}\bar{\mathcal{Q}}^y_k$ is stable under the action of $\mathcal{H}_{k,y}$ on $\bar{\mathcal{Q}}^y_k$. For $k = 1, 2$ set

$${}^{\bar{d}}\bar{\mathcal{Q}}_{k+1,ex} = {}^{\bar{d}}\bar{\mathcal{Q}}_k \times_{\bar{\mathcal{Q}}_k} \bar{\mathcal{Q}}_{k+1,ex}$$

Set also

$${}^{\bar{d}}\bar{\mathcal{Q}}_{0,ex} = {}^{\bar{d}}\bar{\mathcal{Q}}_1 \times_{\bar{\mathcal{Q}}_1} \bar{\mathcal{Q}}_{0,ex}$$

For $y \in X$ denote by ${}^{\bar d}\bar{\mathcal{Q}}^y_{k+1,ex}$ the preimage of $\bar{\mathcal{Q}}^y_{k+1,ex}$ under ${}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex} \to \bar{\mathcal{Q}}_{k+1,ex}$. For $k = 1, 2$ the stack ${}^{\bar d}\bar{\mathcal{Q}}^y_{k+1,ex}$ is stable under the action of $\mathcal{H}_{k,y}$ on $\bar{\mathcal{Q}}^y_{k+1,ex}$. The stack ${}^{\bar d}\bar{\mathcal{Q}}^y_{0,ex}$ is stable under the action of $\mathcal{H}_{0,y}$ on $\bar{\mathcal{Q}}^y_{0,ex}$.

Thus, following the same lines one defines the categories $\mathrm{P}^W({}^{\bar d}\bar{\mathcal{Q}}^y_{k+1,ex})$ and $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}^y_{k+1,ex})$, and further $\mathrm{P}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$ and $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$ for $k = -1, 1, 2$. Similarly for $\mathrm{P}^W({}^{\bar d}\bar{\mathcal{Q}}_2)$ and $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_2)$.

By abuse of notation, write

$$i_0 : {}^{\bar d}\bar{\mathcal{Q}}_1 \hookrightarrow {}^{\bar d}\bar{\mathcal{Q}}_{0,ex}$$

for the natural closed immersion. Denote by $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_1) \subset \mathrm{D}({}^{\bar d}\bar{\mathcal{Q}}_1)$ the full triangulated subcategory consisting of $K \in \mathrm{D}({}^{\bar d}\bar{\mathcal{Q}}_1)$ such that

$$(i_0)_! K \in \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{0,ex})$$

As in ([5], Lemma 4.11) one shows the following (cf. also Appendix A).

Lemma 1. *1) Let $k = -1, 1, 2$. The functors of $*$- and $!$-restriction map $\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ to $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$. The functors of $*$- and $!$-direct image map $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$ to $\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$.*

For $K \in \mathrm{D}(\bar{\mathcal{Q}}_{k+1,ex})$ we have $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ if and only if its $*$-restriction (or, equivalently, $!$-restriction) to ${}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex}$ lies in $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$ for any $\bar d$.
2) Let $k = 1, 2$. For $K \in \mathrm{D}(\bar{\mathcal{Q}}_k)$ we have $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_k)$ if and only if its $*$-restriction (or, equivalently, $!$-restriction) to ${}^{\bar d}\bar{\mathcal{Q}}_k$ lies in $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_k)$ for any $\bar d$.

2.14 For $k = 1, 2$ define a closed substack ${}^{\bar d}\bar{\mathcal{Q}}'_{k+1,ex} \hookrightarrow {}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex}$ by the conditions: t_2 comes from $\mathrm{H}^0(X, \Omega^{-1} \otimes \mathcal{W}(-2D_1))$ in both cases, and for $k = 2$ we require in addition that t_3 comes from $\mathrm{H}^0(X, \mathcal{A} \otimes \Omega(-2D'_2))$, where we have put $D'_2 = D_2 - D_1$.

Let us define for $k = 1, 2$ a natural map

$${}^{\bar d}\chi_{k+1,ex} : {}^{\bar d}\bar{\mathcal{Q}}'_{k+1,ex} \to \mathbb{A}^1$$

CASE $k = 1$. The stack ${}^{\bar d}\bar{\mathcal{Q}}'_{2,ex}$ classifies collections: $D_1 \in X^{(d_1)}$, a P_1-torsor on X given by

$$(L_1 \subset L_{-1} \subset M)$$

with $L_1 \,\tilde{\to}\, \Omega(D_1)$, and a section $s : \mathcal{O}(D_1) \to L_{-1}/L_1$.

The map ${}^{\bar d}\chi_{2,ex}$ sends this collection to the class in $\mathrm{Ext}^1(\mathcal{O}(D_1), \Omega(D_1)) \,\tilde{\to}\, k$ of the pull-back of $0 \to L_1 \to L_2 \to L_2/L_1 \to 0$ under s.

CASE $k = 2$. Note that Bun_P is the stack classifying: a rank 2 vector bundle L_2 on X, a line bundle $\mathcal{A}$ on X, and an exact sequence $0 \to \mathrm{Sym}^2 L_2 \to ? \to \mathcal{A} \to 0$. For such $\mathcal{F}_P \in \mathrm{Bun}_P$ the vector bundle $\mathcal{F}_P \times_P U$ is $(\mathrm{Sym}^2 L_2) \otimes \mathcal{A}^{-1}$.

Therefore, the stack ${}^{\bar d}\bar{\mathcal{Q}}'_{3,ex}$ classifies collections: $D_1\in X^{(d_1)}$, $D'_2\in X^{(d_2-d_1)}$ with $D'_2\ge D_1$, two exact sequences

$$0\to L_1\to L_2\to L_2/L_1\to 0$$

and

$$0\to \operatorname{Sym}^2 L_2\to ?\to \mathcal{A}\to 0 \qquad (6)$$

with $L_1\widetilde{\to}\Omega(D_1)$ and $L_2/L_1\widetilde{\to}\mathcal{O}(D'_2)$, and a section $t_3:\Omega^{-1}(2D'_2)\to\mathcal{A}$.

The map ${}^{\bar d}\chi_{3,ex}$ sends this collection to the sum of two numbers, the first being defined as for ${}^{\bar d}\chi_{2,ex}$, and the second is the class in $\operatorname{Ext}^1(\Omega^{-1}(2D'_2),\mathcal{O}(2D'_2))\widetilde{\to} k$ of the pull-back of

$$0\to \operatorname{Sym}^2(L_2/L_1)\to ?\to\mathcal{A}\to 0 \qquad (7)$$

under t_3. Here (7) is the push-forward of (6) under $\operatorname{Sym}^2 L_2\to\operatorname{Sym}^2(L_2/L_1)$.

2.15 For $k=1,2$ define the stack ${}^{\bar d}\mathcal{P}_{k+1,ex}$ as follows.

The stack ${}^{\bar d}\mathcal{P}_{2,ex}$ classifies: $D_1\in X^{(d_1)}$, a rank 2 vector bundle M_2 on X with section $s:\mathcal{O}(D_1)\to M_2$.

The stack ${}^{\bar d}\mathcal{P}_{3,ex}$ classifies: $D_1\in X^{(d_1)}$, $D'_2\in X^{(d_2-d_1)}$ with $D'_2\ge D_1$, a line bundle $\mathcal{A}$ on X with a section $\Omega^{-1}(2D'_2)\to\mathcal{A}$.

In both cases we have a projection $\phi_{k+1,ex}:{}^{\bar d}\bar{\mathcal{Q}}'_{k+1,ex}\to{}^{\bar d}\mathcal{P}_{k+1,ex}$. For $k=1$ it is given by $M_2=L_{-1}/L_1$.

2.16 For $\bar d=(d_1,d_2)$ let ${}^{\bar d}\bar{\mathcal{Q}}'_2\subset{}^{\bar d}\bar{\mathcal{Q}}_2$ be the closed substack given by $D'_2\ge D_1$, where $D'_2=D_2-D_1$. We have a natural map

$${}^{\bar d}\chi_2:{}^{\bar d}\bar{\mathcal{Q}}'_2\to\mathbb{A}^1$$

defined in the same way as ${}^{\bar d}\chi_{2,ex}$.

For $k=1,2$ and $\bar d=(d_1,\dots,d_k)$ as above define the stack ${}^{\bar d}\mathcal{P}_k$ as follows. The stack ${}^{\bar d}\mathcal{P}_1$ classifies $D_1\in X^{(d_1)}$ and an exact sequence of vector bundles on X

$$0\to L_1\to L_{-1}\to L_{-1}/L_1\to 0$$

with $L_1\widetilde{\to}\Omega(D_1)$, where L_{-1}/L_1 is of rank 2.

The stack ${}^{\bar d}\mathcal{P}_2$ classifies $D_1\in X^{(d_1)}$, $D'_2\in X^{(d_2-d_1)}$ with $D'_2\ge D_1$, a line bundle $\mathcal{A}$ on X, and an exact sequence on X

$$0\to\mathcal{O}(D'_2)\to M_2\to\mathcal{A}(-D'_2)\to 0$$

We have projections $\phi_1:{}^{\bar d}\bar{\mathcal{Q}}_1\to{}^{\bar d}\mathcal{P}_1$ and $\phi_2:{}^{\bar d}\bar{\mathcal{Q}}'_2\to{}^{\bar d}\mathcal{P}_2$.

As in ([5], Proposition 4.13) one proves

Lemma 2. *For $k=1,2$ and a string of nonnegative integers $\bar d=(d_1,\dots,d_k)$ we have the following.*

i) Any object $K \in \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$ *is supported at* ${}^{\bar d}\bar{\mathcal{Q}}'_{k+1,ex}$. *The functor* $K \mapsto {}^{\bar d}\chi^*_{k+1,ex}\mathcal{L}_\psi \otimes \phi^*_{k+1,ex}K$ *provides an equivalence of categories* $\mathrm{D}({}^{\bar d}\mathcal{P}_{k+1,ex}) \to \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$.

ii) We have an equivalence of categories $\mathrm{D}({}^{\bar d}\mathcal{P}_k) \to \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_k)$. *For* $k=1$ *it is given by the functor* $K \mapsto \phi_1^*K$, *whence for* $k=2$ *it is given by the functor* $K \mapsto \phi_2^*K \otimes {}^{\bar d}\chi_2^*\mathcal{L}_\psi$. $\square$

3. Whittaker functors

In this section we prove the following theorem.

Theorem 1. *i) There is an equivalence of categories* $W_{1,0,ex} : \mathrm{D}(\bar{\mathcal{Q}}_1) \to \mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex})$, *which is* t*-exact, and* $(\pi_{0,1,ex})_!$ *is quasi-inverse to it. Moreover, for any* $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex})$ *the natural map* $(\pi_{0,1,ex})_!K \to (\pi_{0,1,ex})_*K$ *is an isomorphism.*

ii) For $k=1,2$ *there is an equivalence of categories* $W_{k,k+1,ex} : \mathrm{D}^W(\bar{\mathcal{Q}}_k) \to \mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$, *which is* t*-exact, and* $(\pi_{k+1,k,ex})_!$ *is quasi-inverse to it. Moreover, for any* $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ *the natural map* $(\pi_{k+1,k,ex})_!K \to (\pi_{k+1,k,ex})_*K$ *is an isomorphism.*

3.1 First, we explain what the corresponding functors do on strata. For $k=1,2$ let $\bar d = (d_1,\dots,d_k)$ be a string of nonnegative integers. Using Lemma 2, define

$$
{}^{\bar d}W_{k,k+1,ex} : \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_k) \to \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})
$$

as the composition

$$
\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_k) \,\widetilde{\to}\, \mathrm{D}({}^{\bar d}\mathcal{P}_k) \stackrel{\mathrm{Four}}{\to} \mathrm{D}({}^{\bar d}\mathcal{P}_{k+1,ex}) \,\widetilde{\to}\, \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})
$$

So, ${}^{\bar d}W_{k,k+1,ex}$ is an equivalence of triangulated categories and t-exact. It also follows from the standard properties of the Fourier transform that $(\pi_{k+1,k,ex})_!$ is quasi-inverse to ${}^{\bar d}W_{k,k+1,ex}$, and we have $(\pi_{k+1,k,ex})_!K \,\widetilde{\to}\, (\pi_{k+1,k,ex})_*K$ for any $K \in \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{k+1,ex})$.

3.2 For $\bar d = (d_1)$ define the functor

$$
{}^{\bar d}W_{1,0,ex} : \mathrm{D}({}^{\bar d}\bar{\mathcal{Q}}_1) \to \mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{0,ex})
$$

as follows. Let ${}^{\bar d}\mathcal{E} \to {}^{\bar d}\bar{\mathcal{Q}}_1$ be the stack whose fibre over a point of ${}^{\bar d}\bar{\mathcal{Q}}_1$ is the stack of exact sequences $0 \to \Omega^2(2D_1)\otimes\mathcal{A}^{-1} \to ? \to \mathcal{O} \to 0$. This is a groupoid over ${}^{\bar d}\bar{\mathcal{Q}}_1$, let ${}^{\bar d}\mathrm{act} : {}^{\bar d}\mathcal{E} \to {}^{\bar d}\bar{\mathcal{Q}}_1$ denote the action.

Let ${}^{\bar d}\bar{\mathcal{Q}}'_{0,ex} \subset {}^{\bar d}\bar{\mathcal{Q}}_{0,ex}$ be the closed substack given by: t_0 comes from $\mathrm{H}^0(X, \mathcal{A}\otimes\Omega^{-1}(-2D_1))$. Any object of $\mathrm{D}^W({}^{\bar d}\bar{\mathcal{Q}}_{0,ex})$ is supported on ${}^{\bar d}\bar{\mathcal{Q}}'_{0,ex}$. As in Appendix A.2, let

$$
{}^{\bar d}W_{1,0,ex}(K) = \mathrm{Four}({}^{\bar d}\mathrm{act}^*\,K)[\dim.\mathrm{rel}](\frac{\dim.\mathrm{rel}}{2}),
$$

where dim. rel is the relative dimension of ${}^{\bar d}\mathcal{E} \to {}^{\bar d}\bar{\mathcal{Q}}_1$. This functor satisfies the same properties as ${}^{\bar d}W_{k,k+1,ex}$ in Sect. 3.1.

3.3 For $k = 1, 2$ we single out the subgroupoids $\mathcal{H}'_{k,y} \subset \mathcal{H}_{k,y}$ as follows.

CASE $k = 1$**.** We let $\mathcal{H}'_{1,y} = \mathcal{H}_{1,y}$ and ${}^i\mathcal{H}'_{1,y} = {}^i\mathcal{H}_{1,y}$. Recall the map $\mathcal{E}_1 \to \bar{\mathcal{Q}}_1^y$ defined in Sect. 2.9. Write $\mathcal{E}_1$ as a union of vector bundles ${}^i\mathcal{E}_1 \to \bar{\mathcal{Q}}_1^y$ with fibre

$$(L_{-1}/\Omega)^* \otimes (\Omega(iy)/\Omega)$$

The fibre of ${}^i\mathcal{E}_1^* \to \bar{\mathcal{Q}}_1^y$ is $(L_{-1}/\Omega) \otimes (\mathcal{O}/\mathcal{O}(-iy))$.

CASE $k = 2$**.** A point of $\bar{\mathcal{Q}}_2^y$ gives rise to a P_2-bundle $\mathcal{F}_{P_2}$ on D_y given by $(L_1 \subset L_2 \subset M)$ with $L_1 \widetilde{\to} \Omega\mid_{D_y}$ and $L_2/L_1 \widetilde{\to} \mathcal{O}\mid_{D_y}$. The P_2-bundle $\mathcal{F}_{P_2}$ gives rise to a P-bundle $\mathcal{F}_P = \mathcal{F}_{P_2} \times_{P_2} P$ on D_y.

By definition, the fibre of $\mathcal{H}'_{2,y} \to \bar{\mathcal{Q}}_2^y$ is

$$\mathrm{H}^0(D_y^*,\ \mathcal{F}_P \times_P U)/\mathrm{H}^0(D_y,\ \mathcal{F}_P \times_P U) \widetilde{\to} (\mathrm{Sym}^2 L_2) \otimes (\mathcal{A}^{-1}(\infty y)/\mathcal{A}^{-1})$$

Let ${}^i\mathcal{H}'_{2,y} \subset \mathcal{H}'_{2,y}$ be the subgroupoid with fibre

$$(\mathrm{Sym}^2 L_2) \otimes (\mathcal{A}^{-1}(iy)/\mathcal{A}^{-1})$$

Let $\mathcal{E}_2 \to \bar{\mathcal{Q}}_2^y$ be the stack with fibre

$$(\mathrm{Sym}^2(L_2/L_1)) \otimes (\mathcal{A}^{-1}(\infty y)/\mathcal{A}^{-1}) \widetilde{\to} \mathcal{A}^{-1}(\infty y)/\mathcal{A}^{-1}$$

This is a union of vector bundles ${}^i\mathcal{E}_2 \to \bar{\mathcal{Q}}_2^y$ with fibre $\mathcal{A}^{-1}(iy)/\mathcal{A}^{-1}$. The fibre of ${}^i\mathcal{E}_2^* \to \bar{\mathcal{Q}}_2^y$ is $\mathcal{A} \otimes (\Omega/\Omega(-iy))$.

3.4 For $k = 1, 2$ we have a natural map $\mathcal{H}'_{k,y} \to \mathcal{E}_k$ over $\bar{\mathcal{Q}}_k^y$. Without loss of generality we may assume that the image of ${}^i\mathcal{H}'_{k,y}$ in $\mathcal{E}_k$ is ${}^i\mathcal{E}_k$. The corresponding map ${}^ip_k : {}^i\mathcal{H}'_{k,y} \to {}^i\mathcal{E}_k$ is smooth with contractible fibres, we denote by $d_{i,k}$ its relative dimension. From ([5], Lemma 4.8) we get

Lemma 3. *The functor $K \mapsto {}^ip_k^*K[d_{i,k}]$ is t-exact and identifies $\mathrm{D}({}^i\mathcal{E}_k)$ with a full triangulated subcategory of $\mathrm{D}({}^i\mathcal{H}'_{k,y})$.* $\square$

For $i' \geq i$ we have ${}^i\mathcal{E}_k \hookrightarrow {}^{i'}\mathcal{E}_k$ is a subbundle. Denote by $\mathrm{pr}_{i',i} : {}^{i'}\mathcal{E}_k^* \to {}^i\mathcal{E}_k^*$ the dual map.

Lemma 4. *For each $i \geq 0$ we have a natural map $f_i : \bar{\mathcal{Q}}_{k+1,ex}^y \to {}^i\mathcal{E}_k^*$ over $\bar{\mathcal{Q}}_k^y$. For $i' \geq i$ the composition*

$$\bar{\mathcal{Q}}_{k+1,ex}^y \xrightarrow{f_{i'}} {}^{i'}\mathcal{E}_k^* \xrightarrow{\mathrm{pr}_{i',i}} {}^i\mathcal{E}_k^*$$

equals f_i. For each open substack of finite type $U \subset \bar{\mathcal{Q}}_k^y$ there is an integer $i(U)$ such that over the preimage of U, the map $f_i : \bar{\mathcal{Q}}_{k+1,ex}^y \to {}^i\mathcal{E}_k^*$ is a closed embedding for every $i \geq i(U)$.

Proof
CASE $k = 1$**.** Given a point of $\bar{\mathcal{Q}}^y_{2,ex}$, over D_y the section $t_2 : \Omega \to \mathcal{W}$ yields a map $s : \mathcal{O} \to L_{-1}/\Omega$ such that $t_2 = t_1 \wedge s$. Now f_i sends a point of $\bar{\mathcal{Q}}^y_{2,ex}$ to the image of s in ${}^i\mathcal{E}^*_1$.

Let ${}^i\mathcal{V} \to \bar{\mathcal{Q}}^y_1$ be the vector bundle with fibre $\mathrm{Hom}(\Omega, \mathcal{W}/\mathcal{W}(-iy))$. Given a point of $\bar{\mathcal{Q}}^y_1$, we have a subbundle $\Omega \otimes (L_{-1}/\Omega) \mid_{D_y} \subset \mathcal{W} \mid_{D_y}$ over D_y. Therefore,

$$(L_{-1}/\Omega) \otimes (\Omega/\Omega(-iy)) \hookrightarrow \mathcal{W}/\mathcal{W}(-iy)$$

So, we have a natural closed embedding ${}^i\mathcal{E}^*_1 \to {}^i\mathcal{V}$ over $\bar{\mathcal{Q}}^y_1$.

Let $i(U)$ be such that the vector space $\mathrm{Hom}(\Omega, \mathcal{W}(-iy))$ is zero for any point of U. Then for $i \geq i(U)$ the natural map $\bar{\mathcal{Q}}^y_{2,ex} \to {}^i\mathcal{V}$ is a closed embedding over U. So, $\bar{\mathcal{Q}}^y_{2,ex} \to {}^i\mathcal{E}^*_1$ is a closed embedding over U for $i \geq i(U)$.

CASE $k = 2$**.** The map f_i sends a point of $\bar{\mathcal{Q}}^y_{3,ex}$ to the image of $t_3 \in \mathrm{H}^0(X, \mathcal{A} \otimes \Omega)$ in $\mathcal{A} \otimes (\Omega/\Omega(-iy))$.

Let $i(U)$ be such that the vector space $\mathrm{H}^0(X, \mathcal{A} \otimes \Omega(-iy))$ is zero for any point of U. Then for $i \geq i(U)$ the map f_i is a closed embedding. $\square$

3.5 For $k = 1, 2$ let ${}^i\mathrm{act}'_k : {}^i\mathcal{H}'_{k,y} \to \bar{\mathcal{Q}}^y_k$ denote the action map. It is smooth, and we denote by $a_{i,k}$ its relative dimension.

Define the functor

$$W^{y,i}_{k,k+1,ex} : \mathrm{D}^W(\bar{\mathcal{Q}}^y_k) \to \mathrm{D}({}^i\mathcal{E}^*_k)$$

as follows. Given $K \in \mathrm{D}^W(\bar{\mathcal{Q}}^y_k)$, from Lemma 3 we learn that there exists $\tilde{K} \in \mathrm{D}({}^i\mathcal{E}_k)$ and an isomorphism

$$h : {}^ip^*_k\tilde{K}[d_{i,k}](\frac{d_{i,k}}{2}) \widetilde{\to} ({}^i\mathrm{act}'_k)^*K[a_{i,k}](\frac{a_{i,k}}{2}) \tag{8}$$

The pair $(\tilde{K}, h)$ is defined up to a unique isomorphism. Set $W^{y,i}_{k,k+1,ex}(K) = \mathrm{Four}(\tilde{K})$.

By construction, the functor $W^{y,i}_{k,k+1,ex}$ is t-exact.

Remark 2. We could replace ${}^i\mathcal{H}'_{k,y}$ by any subgroupoid ${}^i\mathcal{H}'_{k,y} \subset \mathcal{H}'_{k,y}$ of finite type over $\bar{\mathcal{Q}}^y_k$ such that the image of ${}^i\mathcal{H}'_{k,y}$ in $\mathcal{E}_k$ is ${}^i\mathcal{E}_k$. The corresponding functors $W^{y,i}_{k,k+1,ex} : \mathrm{D}^W(\bar{\mathcal{Q}}^y_k) \to \mathrm{D}({}^i\mathcal{E}^*_k)$ would be naturally isomorphic. Thus, the functors $W^{y,i}_{k,k+1,ex}$ do not depend on the choice of the group subschemes ${}^iN_{k,y}$ inside of $N_{k,y}$.

Using the above remark together with appendix A.3, one shows that for $i' \geq i$ we have an isomorphism of functors $(\mathrm{pr}_{i',i})_! \circ W^{y,i'}_{k,k+1,ex} \widetilde{\to} W^{y,i}_{k;k+1,ex}$.

Lemma 5. *For $k = 1, 2$ let $K \in \mathrm{D}^W(\bar{\mathcal{Q}}^y_k)$. For any open substack of finite type $U \subset \bar{\mathcal{Q}}^y_k$ and any integer i large enough (in particular, $i \geq i(U)$ of Lemma 4), over the preimage of U, the complex $W^{y,i}_{k,k+1,ex}(K)$ is supported on $\bar{\mathcal{Q}}^y_{k+1,ex} \subset {}^i\mathcal{E}^*_k$.*

Proof Since U is contained in a finite number of strata ${}^{\bar d}\bar{\mathcal{Q}}_k$, we are easily reduced to the case where $U\subset {}^{\bar d}\bar{\mathcal{Q}}^y_k$ for some $\bar d$, and K is the extension by zero from U.

CASE $k=1$**.** There is $i'(U)$ such that for any point of U given by $D_1\in (X-y)^{(d_1)}$, $(\Omega(D_1)\subset L_{-1}\subset M)\in \mathrm{Bun}_{P_1}$ we have $\mathrm{H}^1(X,\ (L_{-1}/\Omega(D_1))^*\otimes\Omega(D_1+iy))=0$. So, for any $i\ge i'(U)$ the natural map

$$ {}^i\mathcal{E}_1\to \mathrm{Ext}^1(L_{-1}/\Omega(D_1),\Omega(D_1)) $$

is surjective over U. If $i\ge i(U), i'(U)$ then $W^{y,i}_{1,2,ex}(K)$ is supported at ${}^{\bar d}\bar{\mathcal{Q}}^y_{2,ex}$ and is isomorphic to ${}^{\bar d}W_{1,2,ex}(K)$.

CASE $k=2$**.** Recall that a point of U is given by a collection: $D_1\in (X-y)^{(d_1)}$, $D'_2\in (X-y)^{(d_2-d_1)}$ with $D'_2\ge D_1$ and $(L_1\subset L_2\subset L_{-1}\subset M)\in\mathrm{Bun}_{P_2}$ with $L_1\widetilde{\to}\Omega(D_1)$ and $L_2/L_1\widetilde{\to}\mathcal{O}(D'_2)$. There is $i'(U)$ such that for any point of U as above we have

$$ \mathrm{H}^1(X,(\mathrm{Sym}^2(L_2/L_1))\otimes\mathcal{A}^{-1}(iy))=0 $$

This implies that for $i\ge i'(U)$ the natural map

$$ {}^i\mathcal{E}_2\to \mathrm{Ext}^1(\mathcal{A},\,\mathrm{Sym}^2(L_2/L_1)) $$

is surjective over U. If $i\ge i(U), i'(U)$ then $W^{y,i}_{2,3,ex}(K)$ is supported at ${}^{\bar d}\bar{\mathcal{Q}}^y_{3,ex}$ and is isomorphic to ${}^{\bar d}W_{2,3,ex}(K)$. $\square$

Thus, we get a well-defined functor $W^y_{k,k+1,ex}:\mathrm{D}^W(\bar{\mathcal{Q}}^y_k)\to\mathrm{D}(\bar{\mathcal{Q}}^y_{k+1,ex})$, it is t-exact by construction.

Given $K\in\mathrm{D}^W(\bar{\mathcal{Q}}^y_k)$, the $*$-restriction of $W^y_{k,k+1,ex}(K)$ to ${}^{\bar d}\bar{\mathcal{Q}}^y_{k+1,ex}$ is naturally isomorphic to ${}^{\bar d}W_{k,k+1,ex}$ applied to the $*$-restriction $K\mid_{{}^{\bar d}\bar{\mathcal{Q}}^y_k}$. By 1) of Lemma 1, we conclude that the image of $W^y_{k,k+1,ex}$ lies in $\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$.

Proposition 1. *For $k=1,2$ the functor $K\mapsto(\pi_{k+1,k,ex})_!K$ maps $\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$ to $\mathrm{D}^W(\bar{\mathcal{Q}}^y_k)$ and is quasi-inverse to $W^y_{k,k+1,ex}$. Moreover, for $K\in\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$ the natural map $(\pi_{k+1,k,ex})_!K\to(\pi_{k+1,k,ex})_*K$ is an isomorphism.*

Proof First, let us show that for $K\in\mathrm{D}^W(\bar{\mathcal{Q}}^y_k)$ we have $(\pi_{k+1,k,ex})_!W^y_{k,k+1,ex}(K)\widetilde{\to}K$ naturally. Indeed, over an open substack of finite type $U\subset\bar{\mathcal{Q}}^y_k$ and i large enough we have $W^y_{k,k+1,ex}(K)=\mathrm{Four}(\tilde K)$ and

$$ (\pi_{k+1,k,ex})_!W^y_{k,k+1,ex}(K)\widetilde{\to} i_U^*\tilde K[d_{i,k}-a_{i,k}](\frac{d_{i,k}-a_{i,k}}{2}), $$

where $\tilde K$ is that of (8), and $i_U: U\to {}^i\mathcal{E}_k$ is the zero section. The equivariance property of K implies that the RHS of the above formula is identified with $K\mid_U$.

The fact that $K\mapsto(\pi_{k+1,k,ex})_!K$ maps $\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$ to $\mathrm{D}^W(\bar{\mathcal{Q}}^y_k)$ follows from Appendix A.1.

Now let us show that for $K\in\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$ we have

$$ W^y_{k,k+1,ex}(\pi_{k+1,k,ex})_!K\widetilde{\to}K \tag{9} $$

naturally. To establish this isomorphism over the preimage of an open substack of finite type $U \subset \bar{\mathcal{Q}}_k^y$, fix an integer i large enough with respect to U.

The groupoid ${}^i\mathcal{H}'_{k,y} \to \bar{\mathcal{Q}}_k^y$ lifts to ${}^i\mathcal{E}_k^*$ in the sense of A.1. In particular, we have a cartesian square

$$\begin{array}{ccc} {}^i\mathcal{H}'_{k,y} \times_{\bar{\mathcal{Q}}_k^y} {}^i\mathcal{E}_k^* & \stackrel{\mathrm{act}}{\to} & {}^i\mathcal{E}_k^* \\ \downarrow \mathrm{id} \times \pi_{\mathcal{E}} & & \downarrow \pi_{\mathcal{E}} \\ {}^i\mathcal{H}'_{k,y} & \stackrel{{}^i\mathrm{act}'_k}{\to} & \bar{\mathcal{Q}}_k^y, \end{array}$$

where we used the projections to define the fibred product, and $\pi_{\mathcal{E}}$ also denotes the projection. We may start with $K \in \mathrm{D}({}^i\mathcal{E}_k^*)$ that satisfies the equivariance property $\mathrm{act}^* K \widetilde{\to} \mathrm{pr}_2^* K \otimes \chi^*\mathcal{L}_\psi$, where χ is the composition

$$ {}^i\mathcal{H}'_{k,y} \times_{\bar{\mathcal{Q}}_k^y} {}^i\mathcal{E}_k^* \to {}^i\mathcal{E}_k \times_{\bar{\mathcal{Q}}_k^y} {}^i\mathcal{E}_k^* \stackrel{\mathrm{ev}}{\to} \mathbb{A}^1 $$

(Actually, for $k = 2$ the complex K satisfies a stronger equivariance property with respect to the action of $\mathcal{H}_{2,y}$, which we don't need for the moment.)

Looking at one more cartesian square

$$\begin{array}{ccc} {}^i\mathcal{H}'_{k,y} \times_{\bar{\mathcal{Q}}_k^y} {}^i\mathcal{E}_k^* & \to & {}^i\mathcal{E}_k \times_{\bar{\mathcal{Q}}_k^y} {}^i\mathcal{E}_k^* \\ \downarrow \mathrm{id} \times \pi_{\mathcal{E}} & & \downarrow \\ {}^i\mathcal{H}'_{k,y} & \stackrel{{}^ip_k}{\to} & {}^i\mathcal{E}_k \end{array}$$

we obtain

$$ ({}^i\mathrm{act}'_k)^*(\pi_{\mathcal{E}})_! K \widetilde{\to} {}^ip_k^* \,\mathrm{Four}(K)[d_{i,k} - a_{i,k}](\frac{d_{i,k} - a_{i,k}}{2}) $$

We have used the fact that the rank of the vector bundle ${}^i\mathcal{E}_k \to \bar{\mathcal{Q}}_k^y$ is $a_{i,k} - d_{i,k}$. The isomorphism (9) over the preimage of U follows.

The above diagrams also show that

$$ ({}^i\mathrm{act}'_k)^*(\pi_{\mathcal{E}})_! K \widetilde{\to} ({}^i\mathrm{act}'_k)^*(\pi_{\mathcal{E}})_* K, $$

because !- and $*$-Fourier transforms coincide. So, $(\pi_{k+1,k,ex})_! K \to (\pi_{k+1,k,ex})_* K$ is an isomorphism. $\square$

Now arguing as in ([5], 5.11) one finishes the proof of Theorem 1 ii).

3.6 The proof of Theorem 1 i) is similar. First, let ${}^i\mathcal{E}_0 = {}^i\mathcal{H}_{0,y}$. The action map ${}^i\mathrm{act}_0 : {}^i\mathcal{H}_{0,y} \to \bar{\mathcal{Q}}_1^y$ is smooth, denote by $a_{i,0}$ its relative dimension. For $i \geq 0$ define the functors

$$ W^{y,i}_{1,0,ex} : \mathrm{D}(\bar{\mathcal{Q}}_1^y) \to \mathrm{D}({}^i\mathcal{E}_0^*) $$

by $W^{y,i}_{1,0,ex}(K) = \mathrm{Four}({}^i\mathrm{act}_0^* K)[a_{i,0}](\frac{a_{i,0}}{2})$. As in Sect. 3.5, this gives rise to a functor $W^y_{1,0,ex} : \mathrm{D}(\bar{\mathcal{Q}}_1^y) \to \mathrm{D}^W(\bar{\mathcal{Q}}^y_{0,ex})$ and so on. The details are left to the reader. $\square$

4. Cuspidality

4.1 Recall the notion of cuspidality on Bun_G. For a proper parabolic $Q \subset G$ let M_Q be its Levi quotient. We have a diagram of natural maps

$$\mathrm{Bun}_{M_Q} \stackrel{\alpha_Q}{\leftarrow} \mathrm{Bun}_Q \stackrel{\beta_Q}{\to} \mathrm{Bun}_G$$

The constant term functor $\mathrm{CT}_Q : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_{M_Q})$ is defined as $\mathrm{CT}_Q(K) = (\alpha_Q)_! \beta_Q^* K$.

A complex $K \in \mathrm{D}(\mathrm{Bun}_G)$ is *cuspidal* if $\mathrm{CT}_Q(K) = 0$ for any standard proper parabolic $P_2 \subset Q \subset G$. It suffices to check this condition for $Q = P_1$ and $Q = P$.

Denote by $\mathrm{D}_{cusp}(\mathrm{Bun}_G) \subset \mathrm{D}(\mathrm{Bun}_G)$ the full triangulated subcategory consisting of cuspidal objects. Similarly, for a scheme of parameters S, one defines $\mathrm{D}_{cusp}(S \times \mathrm{Bun}_G)$.

4.2 Let us introduce the notion of cuspidality on $\bar{\mathcal{Q}}_k$ for $k = 1, 2, 3$.

The stack Bun_{M_1} classifies pairs: a line bundle L_1 on X and a rank 2 bundle M_2 on X. The projection $\mathrm{Bun}_{P_1} \to \mathrm{Bun}_{M_1}$ sends $(L_1 \subset L_{-1} \subset M) \in \mathrm{Bun}_{P_1}$ to $(L_1, M_2 = L_{-1}/L_1)$.

The stack Bun_M classifies pairs: a line bundle $\mathcal{A}$ on X and a rank 2 bundle L_2 on X. The projection $\mathrm{Bun}_P \to \mathrm{Bun}_M$ sends a collection $(\mathcal{A}, L_2, 0 \to \mathrm{Sym}^2 L_2 \to ? \to \mathcal{A} \to 0)$ to $(\mathcal{A}, L_2)$.

For $k = 1, 2$ consider the natural diagram

$$\begin{array}{ccccc}
\bar{\mathcal{Q}}_k & \stackrel{\beta_P^k}{\leftarrow} & \bar{\mathcal{Q}}_k^P & \stackrel{\alpha_P^k}{\to} & \bar{\mathcal{Q}}_k^M \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \stackrel{\beta_P}{\leftarrow} & \mathrm{Bun}_P & \stackrel{\alpha_P}{\to} & \mathrm{Bun}_M,
\end{array}$$

where the right square is cartesian, and the stack $\bar{\mathcal{Q}}_k^M$ classifies collections: an M-torsor $(\mathcal{A}, L_2)$ on X together with sections $t_1, \ldots, t_k$, where

$$\begin{aligned}
t_1 &: \Omega \hookrightarrow L_2 \\
t_2 &: \Omega \hookrightarrow \wedge^2 L_2
\end{aligned}$$

The constant term functor $\mathrm{CT}_P^{\bar{\mathcal{Q}}_k} : \mathrm{D}(\bar{\mathcal{Q}}_k) \to \mathrm{D}(\bar{\mathcal{Q}}_k^M)$ is defined as $\mathrm{CT}_P^{\bar{\mathcal{Q}}_k}(K) = (\alpha_P^k)_! (\beta_P^k)^* K$.

Consider the natural diagram

$$\begin{array}{ccccc}
\bar{\mathcal{Q}}_1 & \stackrel{\beta_{P_1}^1}{\leftarrow} & \bar{\mathcal{Q}}_1^{P_1} & \stackrel{\alpha_{P_1}^1}{\to} & \bar{\mathcal{Q}}_1^{M_1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \stackrel{\beta_{P_1}}{\leftarrow} & \mathrm{Bun}_{P_1} & \stackrel{\alpha_{P_1}}{\to} & \mathrm{Bun}_{M_1},
\end{array}$$

where the right square is cartesian, and the stack $\bar{\mathcal{Q}}_1^{M_1}$ classifies collections: a M_1-torsor (L_1, M_2) on X together with section $t_1 : \Omega \hookrightarrow L_1$.

The constant term functor $\mathrm{CT}_{P_1}^{\bar{\mathcal{Q}}_1} : \mathrm{D}(\bar{\mathcal{Q}}_1) \to \mathrm{D}(\bar{\mathcal{Q}}_1^{M_1})$ is defined as $\mathrm{CT}_{P_1}^{\bar{\mathcal{Q}}_1}(K) = (\alpha_{P_1}^1)_! (\beta_{P_1}^1)^* K$.

Definition 1. i) An object $K \in \mathrm{D}(\bar{\mathcal{Q}}_1)$ is *cuspidal* if $\mathrm{CT}_P^{\bar{\mathcal{Q}}_1}(K) = 0$ and $\mathrm{CT}_{P_1}^{\bar{\mathcal{Q}}_1}(K) = 0$.
ii) An object $K \in \mathrm{D}(\bar{\mathcal{Q}}_2)$ is *cuspidal* if $\mathrm{CT}_P^{\bar{\mathcal{Q}}_2}(K) = 0$.
iii) Any object $K \in \mathrm{D}(\bar{\mathcal{Q}}_3)$ is *cuspidal*.

4.3 For $k = 1, 2$ denote by $W_{k,k+1} : \mathrm{D}^W(\bar{\mathcal{Q}}_k) \to \mathrm{D}^W(\bar{\mathcal{Q}}_{k+1})$ the functor $W_{k,k+1,ex}$ followed by the restriction to $\bar{\mathcal{Q}}_{k+1} \subset \bar{\mathcal{Q}}_{k+1,ex}$.

Proposition 2. *i) The functor $W_{k,k+1} : \mathrm{D}^W(\bar{\mathcal{Q}}_k) \to \mathrm{D}^W(\bar{\mathcal{Q}}_{k+1})$ maps cuspidal objects to cuspidal.*
ii) If $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_k)$ is cuspidal then the $*$-restriction of $W_{k,k+1,ex}(K)$ to $\bar{\mathcal{Q}}_{k+1,ex} - \bar{\mathcal{Q}}_{k+1}$ vanishes.

Proof ii) Note that $\bar{\mathcal{Q}}_{k+1,ex} - \bar{\mathcal{Q}}_{k+1}$ is isomorphic to $\bar{\mathcal{Q}}_k$, the zero section of the bundle $\pi_{k+1,k,ex} : \bar{\mathcal{Q}}_{k+1,ex} \to \bar{\mathcal{Q}}_k$. We will calculate the $*$-restriction $W_{k,k+1,ex}(K) \mid_{{}^{\bar{d}}\bar{\mathcal{Q}}_k}$ for any stratum ${}^{\bar{d}}\bar{\mathcal{Q}}_k \subset \bar{\mathcal{Q}}_k$.

Let $\psi_1 : {}^{\bar{d}}\bar{\mathcal{Q}}_1 \to \bar{\mathcal{Q}}_1^{M_1}$ be the map that sends $((L_1 \subset L_{-1} \subset M),\ \Omega \stackrel{t_1}{\hookrightarrow} L_1)$ to

$$(M_2 = L_{-1}/L_1,\ \Omega \stackrel{t_1}{\hookrightarrow} L_1)$$

Let $\psi_2 : {}^{\bar{d}}\bar{\mathcal{Q}}_2 \to \bar{\mathcal{Q}}_2^M$ be the map that sends $((L_1 \subset L_2 \subset L_{-1} \subset M),\ \Omega \stackrel{t_1}{\hookrightarrow} L_1,\ \Omega \stackrel{t_2}{\hookrightarrow} \wedge^2 L_2)$ to

$$(\mathcal{A}, L_2,\ \Omega \stackrel{t_1}{\hookrightarrow} L_2,\ \Omega \stackrel{t_2}{\hookrightarrow} \wedge^2 L_2)$$

Using Lemma 2, one shows the following:

- for $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_1)$ we have $W_{1,2,ex}(K) \mid_{{}^{\bar{d}}\bar{\mathcal{Q}}_1} \widetilde{\to} \psi_1^* \mathrm{CT}_{P_1}^{\bar{\mathcal{Q}}_1}(K)$ up to a cohomological shift and a twist;
- for $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_2)$ we have $W_{2,3,ex}(K) \mid_{{}^{\bar{d}}\bar{\mathcal{Q}}_2} \widetilde{\to} \psi_2^* \mathrm{CT}_{P}^{\bar{\mathcal{Q}}_2}(K)$ up to a cohomological shift and a twist.

Part ii) follows.

Remark 3. Actually, we showed that for $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_k)$ the condition $W_{k,k+1,ex}(K) \mid_{\bar{\mathcal{Q}}_k} = 0$ is equivalent to $\mathrm{CT}_{P_1}^{\bar{\mathcal{Q}}_1}(K) = 0$ for $k = 1$ (resp., to $\mathrm{CT}_P^{\bar{\mathcal{Q}}_2}(K) = 0$ for $k = 2$). Indeed, as $\bar{d}$ ranges over strings of nonnegative integers $\bar{d} = (d_1, \dots, d_k)$, the images of ψ_k form a stratification of the corresponding stack.

i) Let $\bar{\mathcal{Q}}_{2,ex}^M$ denote the stack classifying $(\mathcal{A}, L_2) \in \mathrm{Bun}_M$ and sections $\Omega \stackrel{t_1}{\hookrightarrow} L_2,\ \Omega \stackrel{t_2}{\to} \wedge^2 L_2$. As in Sect. 4.2, we have the diagram

$$\begin{array}{ccccc}
\bar{\mathcal{Q}}_{2,ex} & \stackrel{\beta_P^{2,ex}}{\leftarrow} & \bar{\mathcal{Q}}_{2,ex}^P & \stackrel{\alpha_P^{2,ex}}{\to} & \bar{\mathcal{Q}}_{2,ex}^M \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \stackrel{\beta_P}{\leftarrow} & \mathrm{Bun}_P & \stackrel{\alpha_P}{\to} & \mathrm{Bun}_M,
\end{array}$$

where the right square is cartesian. Let $\mathrm{CT}_P^{\bar{\mathcal{Q}}_{2,ex}} : \mathrm{D}(\bar{\mathcal{Q}}_{2,ex}) \to \mathrm{D}(\bar{\mathcal{Q}}_{2,ex}^M)$ denote the functor $K \mapsto (\alpha_P^{2,ex})_!(\beta_P^{2,ex})^* K$. Proceeding as in Sect. 2-3, one introduces the category $\mathrm{D}^W(\bar{\mathcal{Q}}_{2,ex}^M)$ and the functor

$$W_{1,2,ex}^M : \mathrm{D}(\bar{\mathcal{Q}}_1^M) \to \mathrm{D}^W(\bar{\mathcal{Q}}_{2,ex}^M),$$

which is also an equivalence of categories.

One checks that $\mathrm{CT}_P^{\bar{\mathcal{Q}}_{2,ex}}$ sends $\mathrm{D}^W(\bar{\mathcal{Q}}_{2,ex})$ to $\mathrm{D}^W(\bar{\mathcal{Q}}^M_{2,ex})$. Let us only indicate that the groupoid $\mathcal{H}_{1,y}\times_{\bar{\mathcal{Q}}_1^y}\bar{\mathcal{Q}}^y_{2,ex}\to\bar{\mathcal{Q}}^y_{2,ex}$ lifts to

$$\bar{\mathcal{Q}}^{P,y}_{2,ex}=\bar{\mathcal{Q}}^P_{2,ex}\times_{\bar{\mathcal{Q}}_{2,ex}}\bar{\mathcal{Q}}^y_{2,ex}$$

We claim that there is a natural isomorphism of functors from $\mathrm{D}^W(\bar{\mathcal{Q}}_1)$ to $\mathrm{D}^W(\bar{\mathcal{Q}}^M_{2,ex})$

$$\mathrm{CT}_P^{\bar{\mathcal{Q}}_{2,ex}}\circ W_{1,2,ex}\,\widetilde{\to}\,W^M_{1,2,ex}\circ\mathrm{CT}_P^{\bar{\mathcal{Q}}_1}\tag{10}$$

The functor $\mathrm{CT}_P^{\bar{\mathcal{Q}}_1}$ admits a right adjoint, which will be denoted by $\mathrm{Eis}_P^{\bar{\mathcal{Q}}_1}$, it sends K to $(\beta^1_P)_*(\alpha^1_P)^!K$. Actually, $\mathrm{CT}_P^{\bar{\mathcal{Q}}_1}$ maps $\mathrm{D}(\bar{\mathcal{Q}}_1^M)$ to $\mathrm{D}^W(\bar{\mathcal{Q}}_1)$.

Similarly, $\mathrm{CT}_P^{\bar{\mathcal{Q}}_{2,ex}}$ admits a right adjoint functor

$$\mathrm{Eis}_P^{\bar{\mathcal{Q}}_{2,ex}}:\mathrm{D}^W(\bar{\mathcal{Q}}^M_{2,ex})\to\mathrm{D}^W(\bar{\mathcal{Q}}_{2,ex})$$

that sends K to $(\beta_P^{2,ex})_*(\alpha_P^{2,ex})^!K$.

We have the following diagram, where the right square is cartesian

$$\begin{array}{ccccc}
\bar{\mathcal{Q}}_1 & \stackrel{\beta^1_P}{\leftarrow} & \bar{\mathcal{Q}}_1^P & \stackrel{\alpha^1_P}{\to} & \bar{\mathcal{Q}}_1^M\\
\uparrow \pi_{2,1,ex} & & \uparrow & & \uparrow \pi^M_{2,1,ex}\\
\bar{\mathcal{Q}}_{2,ex} & \stackrel{\beta^{2,ex}_P}{\leftarrow} & \bar{\mathcal{Q}}^P_{2,ex} & \stackrel{\alpha^{2,ex}_P}{\to} & \bar{\mathcal{Q}}^M_{2,ex}
\end{array}$$

It follows that $(\pi_{2,1,ex})_*\circ\mathrm{Eis}_P^{\bar{\mathcal{Q}}_{2,ex}}\,\widetilde{\to}\,\mathrm{Eis}_P^{\bar{\mathcal{Q}}_1}\circ(\pi^M_{2,1,ex})_*$ naturally. Passing to left adjoint functors, we get the isomorphism (10).

So, if $K\in\mathrm{D}^W(\bar{\mathcal{Q}}_1)$ is cuspidal then $\mathrm{CT}_P^{\bar{\mathcal{Q}}_{2,ex}}W_{1,2,ex}(K)=0$. By i), the complex $W_{1,2,ex}(K)$ is the extension by zero from $\bar{\mathcal{Q}}_2$, so $\mathrm{CT}_P^{\bar{\mathcal{Q}}_2}W_{1,2}(K)=0$ and $W_{1,2}(K)$ is cuspidal. $\square$

Recall the notation $\bar{\mathcal{Q}}=\bar{\mathcal{Q}}_3$. Let $W:\mathrm{D}^W(\bar{\mathcal{Q}}_1)\to\mathrm{D}^W(\bar{\mathcal{Q}})$ be the functor $W_{2,3}\circ W_{1,2}$. Exactly as in ([5], Theorem 6.4), one derives from Proposition 2 the following corolary.

Corolary 1. *For $k=1,2$ let $K_1,K_2\in\mathrm{D}^W(\bar{\mathcal{Q}}_k)$ be two objects with K_1 cuspidal. Then the map $\mathrm{Hom}_{\mathrm{D}^W(\bar{\mathcal{Q}}_k)}(K_1,K_2)\to\mathrm{Hom}_{\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1})}(W_{k,k+1}(K_1),\ W_{k,k+1}(K_2))$ is an isomorphism. So, for $k=1$*

$$\mathrm{Hom}_{\mathrm{D}^W(\bar{\mathcal{Q}}_1)}(K_1,K_2)\to\mathrm{Hom}_{\mathrm{D}^W(\bar{\mathcal{Q}})}(W(K_1),\ W(K_2))$$

is also an isomorphism. $\square$

4.4 We also have the following analog of ([5], Theorem 6.9). For $k=1,2,3$ let $\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_k)\subset\mathrm{D}^W(\bar{\mathcal{Q}}_k)$ denote the full subcategory consisting of cuspidal objects. This is a triangulated subcategory.

Theorem 2. *For $k=1,2$ the functor $W_{k,k+1}$ induces an equivalence of triangulated categories $\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_k)\to \mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_{k+1})$. In particular, $W:\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1)\to \mathrm{D}^W(\bar{\mathcal{Q}})$ is an equivalence.*

Proof We know by Proposition 2 that $W_{k,k+1}$ maps cuspidal objects to cuspidal. Let

$$W^{-1}_{k,k+1}:\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_{k+1})\to \mathrm{D}^W(\bar{\mathcal{Q}}_k)$$

be the functor sending K to $(\pi_{k+1,k,ex})_!K'$, where K' is the extension by zero of K to $\bar{\mathcal{Q}}_{k+1,ex}$.

If $K\in \mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_{k+1})$ then the complex $W^{-1}_{k,k+1}(K)$ is cuspidal. Indeed, for $k=2$ the assertion follows from Remark 3. For $k=1$ set $F=W^{-1}_{1,2}(K)$. We have $\mathrm{CT}^{\bar{\mathcal{Q}}_1}_{P_1}F=0$ by Remark 3. Further, $W^M_{1,2,ex}\mathrm{CT}^{\bar{\mathcal{Q}}_1}_P(F)=0$ by (10). Since the functor $W^M_{1,2,ex}$ is an equivalence, we get $\mathrm{CT}^{\bar{\mathcal{Q}}_1}_P(F)=0$.

Let us show that $W^{-1}_{k,k+1}:\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_{k+1})\to \mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_k)$ is quasi-inverse to $W_{k,k+1}$. From ii) of Theorem 1 we conclude that $W_{k,k+1}\circ W^{-1}_{k,k+1}\widetilde{\to}\mathrm{id}_{\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_{k+1})}$ naturally, and there is a natural adjunction map $W^{-1}_{k,k+1}\circ W_{k,k+1}\to \mathrm{id}_{\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_k)}$.

For $K\in \mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_k)$ consider a distinguished triangle

$$W^{-1}_{k,k+1}W_{k,k+1}(K)\to K\to K'$$

We have $W_{k,k+1}(K')=0$ and K' is cuspidal. Hence, $K'=0$ by Corolary 1. $\square$

5. Hecke functors

5.1 Recall the Hecke functors $\mathrm{H},\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}$ and $\mathrm{H}^{\bar{\mathcal{Q}}_k}$ introduced in Sect. 2.2-2.4.

Proposition 3. *The functor $\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}$ sends $\mathrm{D}^W(\bar{\mathcal{Q}}_{k,ex})$ to $\mathrm{D}^W(X\times\bar{\mathcal{Q}}_{k,ex})$. The functor $\mathrm{H}^{\bar{\mathcal{Q}}_k}$ sends $\mathrm{D}^W(\bar{\mathcal{Q}}_k)$ to $\mathrm{D}^W(X\times\bar{\mathcal{Q}}_k)$.*

Proof Let ${}_x\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}$ denote the functor $\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}$ followed by $*$-restriction to $x\times\bar{\mathcal{Q}}_{k,ex}\subset X\times\bar{\mathcal{Q}}_{k,ex}$. To simplify the notation, we will show that ${}_x\mathrm{H}^{\bar{\mathcal{Q}}_{k+1,ex}}$ preserves the category $\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ for $k=1,2$. The other cases are treated similarly.

Let $y\in X$ be distinct from x. Let ${}_x\mathcal{H}_G$ be the preimage of x under $\mathrm{supp}:\mathcal{H}_G\to X$. We have a well-defined functor ${}_x\mathrm{H}^{\bar{\mathcal{Q}}^y_{k+1,ex}}:\mathrm{D}(\bar{\mathcal{Q}}^y_{k+1,ex})\to\mathrm{D}(\bar{\mathcal{Q}}^y_{k+1,ex})$. Let us show that it preserves the subcategory $\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k+1,ex})$. Indeed, the groupoid

$$\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex}\to\bar{\mathcal{Q}}^y_{k+1,ex}$$

lifts to $\bar{\mathcal{Q}}^y_{k+1,ex}\times_{\mathrm{Bun}_G}{}_x\mathcal{H}_G$ with respect to both $\mathfrak{p}_{k+1,ex}$ and $\mathfrak{q}_{k+1,ex}$, so that we have diagrams

$$\begin{array}{ccccc}
\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex} & \stackrel{\mathfrak{p}_{\mathcal{Z}}}{\leftarrow} & \mathcal{Z} & \stackrel{\mathfrak{q}_{\mathcal{Z}}}{\to} & \mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex}\\
\downarrow \mathrm{pr} & & \downarrow \mathrm{pr} & & \downarrow \mathrm{pr}\\
\bar{\mathcal{Q}}^y_{k+1,ex} & \stackrel{\mathfrak{p}_{k+1,ex}}{\leftarrow} & \bar{\mathcal{Q}}_{k+1,ex}\times_{\mathrm{Bun}_G}{}_x\mathcal{H}_G & \stackrel{\mathfrak{q}_{k+1,ex}}{\to} & \mathcal{Q}^y_{k+1,ex}
\end{array}$$

and

$$
\begin{array}{ccccc}
\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex} & \stackrel{\mathfrak{p}_{\mathcal{Z}}}{\leftarrow} & \mathcal{Z} & \stackrel{\mathfrak{q}_{\mathcal{Z}}}{\rightarrow} & \mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex}\\
\downarrow \mathrm{act} && \downarrow \mathrm{act} && \downarrow \mathrm{act}\\
\bar{\mathcal{Q}}^y_{k+1,ex} & \stackrel{\mathfrak{p}_{k+1,ex}}{\leftarrow} & \bar{\mathcal{Q}}_{k+1,ex}\times_{\mathrm{Bun}_G}{}_x\mathcal{H}_G & \stackrel{\mathfrak{q}_{k+1,ex}}{\rightarrow} & \mathcal{Q}^y_{k+1,ex}
\end{array}
$$

in both of which both squares are cartesian. Moreover, the compositions

$$\mathcal{Z}\stackrel{\mathfrak{p}_{\mathcal{Z}}}{\to}\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex}\stackrel{\chi_{k,y}}{\to}\mathbb{A}^1$$

and

$$\mathcal{Z}\stackrel{\mathfrak{q}_{\mathcal{Z}}}{\to}\mathcal{H}_{k,y}\times_{\bar{\mathcal{Q}}^y_k}\bar{\mathcal{Q}}^y_{k+1,ex}\stackrel{\chi_{k,y}}{\to}\mathbb{A}^1$$

coincide. Thus, ${}_x\mathrm{H}^{\bar{\mathcal{Q}}^y_{k+1,ex}}$ preserves the equivariance condition. Using the following remark, one finishes the proof.

Remark 4. Let $x_1,\dots,x_n$ be a finite collection of points of X. Let $K\in\mathrm{D}(\bar{\mathcal{Q}}_{k,ex})$ be such that its restriction to $\bar{\mathcal{Q}}^y_{k,ex}$ lies in $\mathrm{D}^W(\bar{\mathcal{Q}}^y_{k,ex})$ for any $y\neq x_i$. Then $K\in\mathrm{D}^W(\bar{\mathcal{Q}}_{k,ex})$. Indeed, as y ranges over points of X different from x_i, the union of $\bar{\mathcal{Q}}^y_{k,ex}$ is $\bar{\mathcal{Q}}_{k,ex}$. Similar statement for $\mathrm{D}^W(\bar{\mathcal{Q}}_k)$ holds.

$\square$(Proposition 3)

5.2 Similarly to the GL_n case, Hecke functors and Whittaker functors commute with each other. The proof of the following result mimics that of ([5], Proposition 7.6).

Proposition 4. *i) For $k=1,2$ there is a natural isomorphism of functors*

$$\mathrm{H}^{\bar{\mathcal{Q}}_{k+1,ex}}\circ W_{k,k+1,ex}\widetilde{\to}(\mathrm{id}\times W_{k,k+1,ex})\circ\mathrm{H}^{\bar{\mathcal{Q}}_k}:\ \mathrm{D}^W(\bar{\mathcal{Q}}_k)\to\mathrm{D}^W(X\times\bar{\mathcal{Q}}_{k+1,ex})$$

ii) There is a natural isomorphism of functors

$$\mathrm{H}^{\bar{\mathcal{Q}}_{0,ex}}\circ W_{1,0,ex}\widetilde{\to}(\mathrm{id}\times W_{1,0,ex})\circ\mathrm{H}^{\bar{\mathcal{Q}}_1}:\ \mathrm{D}(\bar{\mathcal{Q}}_1)\to\mathrm{D}^W(X\times\bar{\mathcal{Q}}_{0,ex})$$

Proof i) To simplify the notation, we replace the functors $\mathrm{H}^{\bar{\mathcal{Q}}_k},\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}$ by ${}_x\mathrm{H}^{\bar{\mathcal{Q}}_k}$, ${}_x\mathrm{H}^{\bar{\mathcal{Q}}_{k,ex}}$. In view of Theorem 1, it suffices to show that for $K\in\mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ we have

$${}_x\mathrm{H}^{\bar{\mathcal{Q}}_k}((\pi_{k+1,k,ex})_!K)\widetilde{\to}(\pi_{k+1,k,ex})_!{}_x\mathrm{H}^{\bar{\mathcal{Q}}_{k+1,ex}}(K)$$

For $x\in X$ let $\bar{\mathcal{Q}}_{k+1,ex,x}$ be the stack defined in the same way as $\bar{\mathcal{Q}}_{k+1,ex}$ with the difference that the last map t_{k+1} is allowed to have a pole of order 2 at x for $k=1$ (resp., of order 1 at x for $k=2$).

Write ${}_x\mathcal{H}^{\bar{\mathcal{Q}}_k}$ for the preimage of $x\times\bar{\mathcal{Q}}_k$ under $\mathrm{supp}\times\mathfrak{p}_k:\bar{\mathcal{Q}}_k\times_{\mathrm{Bun}_G}\mathcal{H}_G\to X\times\bar{\mathcal{Q}}_k$. We have a diagram

$$
\begin{array}{ccccc}
\bar{\mathcal{Q}}_{k+1,ex,x} & \stackrel{\mathfrak{p}_{k+1,ex,x}}{\leftarrow} & {}_x\mathcal{H}^{\bar{\mathcal{Q}}_{k+1,ex,x}} & \stackrel{\mathfrak{q}_{k+1,ex}}{\rightarrow} & \bar{\mathcal{Q}}_{k+1,ex}\\
\downarrow && \downarrow && \downarrow \pi_{k+1,k,ex}\\
\bar{\mathcal{Q}}_k & \stackrel{\mathfrak{p}_k}{\leftarrow} & {}_x\mathcal{H}^{\bar{\mathcal{Q}}_k} & \stackrel{\mathfrak{q}_k}{\rightarrow} & \bar{\mathcal{Q}}_k,
\end{array}
$$

where the stack ${}_x\mathcal{H}^{\bar{\mathcal{Q}}_{k+1,ex,x}}$ is defined by the condition that the right square is cartesian, and $\mathfrak{p}_{k+1,ex,x}$ is the natural map.

It suffices to show that for $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_{k+1,ex})$ the complex $(\mathfrak{p}_{k+1,ex,x})_!\mathfrak{q}^*_{k+1,ex}K$ is supported on $\bar{\mathcal{Q}}_{k+1,ex} \subset \bar{\mathcal{Q}}_{k+1,ex,x}$. This direct image will verify an appropriate equivariance condition on $\bar{\mathcal{Q}}_{k+1,ex,x}$. So, our assertion is verified stratum by stratum using an analog of Lemma 2.

Part ii) is proved similarly. $\square$

6. Hyper-cuspidality

6.1 Recall that U_0 denotes the center of U_1. Set $P_0 = P_1/U_0$. We have a diagram of natural maps $\mathrm{Bun}_{P_0} \stackrel{\alpha_{P_0}}{\leftarrow} \mathrm{Bun}_{P_1} \stackrel{\beta_{P_1}}{\to} \mathrm{Bun}_G$. Define the constant term functor

$$\mathrm{CT}_{P_0} : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_{P_0})$$

as $\mathrm{CT}_{P_0}(K) = (\alpha_{P_0})_!\beta^*_{P_1}(K)$. The following is a geometric version of ([9], Definition on p. 328).

Definition 2. i) A complex $K \in \mathrm{D}(\mathrm{Bun}_G)$ is *hyper-cuspidal* if $\mathrm{CT}_{P_0}(K) = 0$.
ii) A complex $K \in \mathrm{D}(\bar{\mathcal{Q}}_1)$ is *hyper-cuspidal* if $W_{1,0,ex}(K)$ is the extension by zero from $\bar{\mathcal{Q}}_0$.

Denote by $\mathrm{D}_{hcusp}(\mathrm{Bun}_G) \subset \mathrm{D}(\mathrm{Bun}_G)$ and by $\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}(\bar{\mathcal{Q}}_1)$ the full triangulated subcategories consisting of hyper-cuspidal objects. Similarly, we have $\mathrm{D}_{hcusp}(S \times \mathrm{Bun}_G)$ for a scheme of parameters S.

If $f : S_1 \to S_2$ is a morphism of schemes then for the map $f \times \mathrm{id} : S_1 \times \mathrm{Bun}_G \to S_2 \times \mathrm{Bun}_G$ the functors $(f \times \mathrm{id})_!$ and $(f \times \mathrm{id})^*$ preserve hyper-cuspidality (and cuspidality). The same is true for the functor $\mathrm{D}(S) \times \mathrm{D}(S \times \mathrm{Bun}_G) \to \mathrm{D}(S \times \mathrm{Bun}_G)$ of the tensor product along S.

Proposition 5. *In both cases* $\mathrm{D}_{hcusp}(\mathrm{Bun}_G) \subset \mathrm{D}_{cusp}(\mathrm{Bun}_G)$ *and* $\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \subset \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)$ *is a full triangulated subcategory.*

Proof Let $K \in \mathrm{D}_{hcusp}(\mathrm{Bun}_G)$. It is clear that $\mathrm{CT}_{P_1}(K) = 0$. Let us show that $\mathrm{CT}_P(K) = 0$. We have a diagram

$$\begin{array}{ccccc} \mathrm{Bun}_G & \leftarrow & \mathrm{Bun}_P & \leftarrow & \mathrm{Bun}_{P_2} \\ & & \downarrow & & \downarrow \\ & & \mathrm{Bun}_M & \leftarrow & \mathrm{Bun}_{B(M)}, \end{array}$$

where $B(M) \subset M$ is a Borel subgroup, the square is cartesian, and the composition in the top line is β_{P_2}. The right vertical arrow factors as $\mathrm{Bun}_{P_2} \stackrel{\delta}{\to} \mathrm{Bun}_{P_2/U_0} \to \mathrm{Bun}_{B(M)}$. So, it is enough to show that $\delta_!\beta^*_{P_2}(K) = 0$.

Since we have the following diagram, where the square is cartesian

$$\begin{array}{ccccc} \mathrm{Bun}_G & \leftarrow & \mathrm{Bun}_{P_1} & \leftarrow & \mathrm{Bun}_{P_2} \\ & & \downarrow & & \downarrow \delta \\ & & \mathrm{Bun}_{P_0} & \leftarrow & \mathrm{Bun}_{P_2/U_0}, \end{array}$$

the first assertion follows.

For sheaves on $\bar{\mathcal{Q}}_1$ the proof is similar. $\square$

6.2 Recall that for each dominant coweight λ of G we have the Hecke functor $\mathrm{H}^{\lambda}_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_G)$ normalized to commute with Verdier duality (cf. [2], Sect. 2.1.4 for the precise definition). In our notation $\mathrm{H}^{\gamma}_G = \mathrm{H}$. It is well-known that the subcategory $\mathrm{D}_{cusp}(\mathrm{Bun}_G) \subset \mathrm{D}(\mathrm{Bun}_G)$ is preserved by Hecke functors. That is, each H^{λ}_G sends $\mathrm{D}_{cusp}(\mathrm{Bun}_G)$ to the category $\mathrm{D}_{cusp}(X \times \mathrm{Bun}_G)$.

Proposition 6. *The subcategory* $\mathrm{D}_{hcusp}(\mathrm{Bun}_G) \subset \mathrm{D}(\mathrm{Bun}_G)$ *is preserved by Hecke functors.*

Proof

Step 1. Let us show that H preserves $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$. One may introduce a version of stacks $\bar{\mathcal{Q}}_1$ and $\bar{\mathcal{Q}}_{0,ex}$, where instead of a fixed T-torsor with trivial conductor $(\mathcal{F}_T, \tilde{\omega})$ one considers all of them as additional parameter. In other words, the stack $\bar{\mathcal{Q}}_1$ would classify $\mathcal{F}_G \in \mathrm{Bun}_G$, a line bundle $\mathcal{B}$ on X and a section $t_1 : \mathcal{B} \hookrightarrow M$; the stack $\bar{\mathcal{Q}}_{0,ex}$ would classify the data just above together with $\mathcal{B}^2 \otimes \Omega^{-1} \to \mathcal{A}$.

An analog of Theorem 1 would hold in this setting. Then for $K \in \mathrm{D}(\mathrm{Bun}_G)$ hyper-cuspidality would be equivalent to requiring that $W_{1,0,ex}(\alpha^* K)$ is the extension by zero from $\bar{\mathcal{Q}}_0$. Our assertion follows from an analog of Proposition 4 ii) in this situation.

Step 2. Recall that Hecke functors can be composed in the following way. For G-dominant coweights λ_1, λ_2 the functor

$$\mathrm{H}^{\lambda_1}_G \star \mathrm{H}^{\lambda_2}_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_G)$$

is defined as

$$\mathrm{H}^{\lambda_1}_G \star \mathrm{H}^{\lambda_2}_G(K) = (\triangle_X^* \boxtimes \mathrm{id})((\mathrm{id} \boxtimes \mathrm{H}^{\lambda_1}) \circ \mathrm{H}^{\lambda_2}_G(K))[-1](\frac{-1}{2})$$

It is known ([2] Sect. 2.1.6 and [1]) that there is a canonical isomorphism functorial in K

$$\mathrm{H}^{\lambda_1}_G \star \mathrm{H}^{\lambda_2}_G(K) \widetilde{\to} \underset{\lambda \in \Lambda^+}{\oplus} \mathrm{H}^{\lambda}_G(K) \otimes \mathrm{Hom}_{\check{G}}(V^{\lambda}, V^{\lambda_1} \otimes V^{\lambda_2})$$

The group of coweight of G orthogonal to all roots is free abelian of rank 1. It is easy to see that Hecke functors corresponding to both generators $\pm\omega$ of this group preserve $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$. One checks that any irreducible representation V^{λ} of $\check{G}$ appears in $(V^{\gamma})^{\otimes k} \otimes V^{r\omega}$ for some $k \geq 0$ and $r \in \mathbb{Z}$. Thus, our assertion follows from the fact that the subcategory $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ is saturated: a direct summand of an object of $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ is again an object of $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$. $\square$

Clearly, the functor $\mathrm{H}^{\bar{\mathcal{Q}}_1}$ preserves the subcategory $\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$, and α^* sends $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)$ to $\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$.

Proposition 7. *We have equivalences of triangulated categories*

i) $\mathrm{D}(\bar{\mathcal{Q}}_1) / \mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \widetilde{\to} \mathrm{D}^W(\bar{\mathcal{Q}}_1)$

ii) $\mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1) / \mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \widetilde{\to} \mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1)$.

Remark 5. Let $F : D \to D'$ be a triangulated functor between triangulated categories. If F admits a fully faithfull right adjoint functor $F' : D' \to D$ then F induces an equivalence of triangulated categories $D/\operatorname{Ker} F \to D'$.

Proof of Proposition 7
i) Recall the closed immersion $i_0 : \bar{\mathcal{Q}}_1 \hookrightarrow \bar{\mathcal{Q}}_{0,ex}$ and its complement $j : \bar{\mathcal{Q}}_0 \hookrightarrow \bar{\mathcal{Q}}_{0,ex}$. The functor $i_0^* : \mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex}) \to \mathrm{D}^W(\bar{\mathcal{Q}}_1)$ adimts a right adjoint $(i_0)_*$, which is fully faithfull. The category $\mathrm{D}^W(\bar{\mathcal{Q}}_0)$ is embedded in $\mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex})$ fully faithfully by $j_!$. By Remark 5, i^* induces an equivalence of triangulated categories

$$\mathrm{D}^W(\bar{\mathcal{Q}}_{0,ex})/\,\mathrm{D}^W(\bar{\mathcal{Q}}_0) \widetilde{\to} \mathrm{D}^W(\bar{\mathcal{Q}}_1)$$

So, the functor $i_0^* \circ W_{0,1,ex}$ induces an equivalence i).
ii) For $K \in \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)$ let us show that $i_0^* W_{0,1,ex}(K)$ is cuspidal. We have a distinguished triangle in $\mathrm{D}(\bar{\mathcal{Q}}_1)$

$$(\pi_{0,1})_! j^* W_{0,1,ex}(K) \to K \to i_0^* W_{0,1,ex}(K)$$

Since $(\pi_{0,1})_! j^* W_{0,1,ex}(K)$ is hyper-cuspidal, it is cuspidal by Proposition 5. So, $i_0^* W_{0,1,ex}(K)$ is also cuspidal.

We conclude that $i_0^* \circ W_{0,1,ex}$ induces a functor $F : \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\,\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1) \to \mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1)$. Let F^{-1} denote the composition

$$\mathrm{D}^W_{cusp}(\bar{\mathcal{Q}}_1) \to \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1) \to \mathrm{D}_{cusp}(\bar{\mathcal{Q}}_1)/\,\mathrm{D}_{hcusp}(\bar{\mathcal{Q}}_1)$$

We claim that F and F^{-1} are quasi-inverse to each other. Indeed, the above distinguished triangle shows that $\mathrm{id} \widetilde{\to} F^{-1} \circ F$. Since for $K \in \mathrm{D}^W(\bar{\mathcal{Q}}_1)$ we have $W_{0,1,ex}(K) \widetilde{\to} (i_0)_* K$ naturally, it follows that $\mathrm{id} \widetilde{\to} F \circ F^{-1}$. $\square$

6.3 If D' is a triangulated category and $D \subset D'$ is a full triangulated subcategory, we write $D^\perp \subset D'$ for the full subcategory consisting of $K \in D'$ such that $\operatorname{Hom}_{D'}(L, K) = 0$ for all $L \in D$. Then $D^\perp \subset D'$ is a full triangulated subcategory, and the composition $D^\perp \to D' \to D'/D$ is fully faithfull (cf. [11], Proposition 2.3.3, p.128).

Consider the subcategory $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)^\perp \subset \mathrm{D}_{cusp}(\mathrm{Bun}_G)$. Let ${}_x\mathrm{H}^\lambda_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)$ denote the functor H^λ_G followed by $*$-restriction to $x \times \mathrm{Bun}_G \hookrightarrow X \times \mathrm{Bun}_G$. Since Hecke functors admit left and right adjoint functors (cf.[2], 3.2.4), it follows that $\mathrm{D}_{hcusp}(\mathrm{Bun}_G)^\perp$ is preserved by all functors ${}_x\mathrm{H}^\lambda_G$.

7. More Whittaker type functors

7.1 Let $\mathcal{Z}_1$ be the stack of collections: $(M, \mathcal{A}) \in \mathrm{Bun}_G$ together with an isotropic subsheaf $L_2 \subset M$, where $L_2 \in \mathrm{Bun}_2$. The stack $\mathcal{Z}_1$ is nothing but $\widetilde{\mathrm{Bun}}_P$ in the notation of ([2], 1.3.6).

Let $\pi_{2,1,ex} : \mathcal{Z}_{2,ex} \to \mathcal{Z}_1$ be the stack over $\mathcal{Z}_1$ with fibre consisting of all maps

$$s : \Omega^{-1} \to \mathcal{A} \otimes \operatorname{Sym}^2 L_2^* \tag{11}$$

Let $\pi_{2,1}:\mathcal{Z}_2\to\mathcal{Z}_1$ be the open substack of $\mathcal{Z}_{2,ex}$ given by the condition: s is injective.

For $k=1,2$ we have the diagram

$$\mathcal{Z}_k \stackrel{\mathfrak{p}_k}{\leftarrow} \mathcal{Z}_k\times_{\mathrm{Bun}_G}\mathcal{H}_G \stackrel{\mathfrak{q}_k}{\to} \mathcal{Z}_k,$$

where we used the map $\mathfrak{p}:\mathcal{H}_G\to\mathrm{Bun}_G$ in the definition of the fibred product, $\mathfrak{p}_k$ is the projection, and $\mathfrak{q}_k$ sends a point of $\mathcal{Z}_k\times_{\mathrm{Bun}_G}\mathcal{H}_G$ to $\mathcal{F}'_G$ equiped with an isotropic subsheaf (and for $k=2$ a section s') that are the compositions

$$\begin{array}{l} L_2\hookrightarrow M\hookrightarrow M' \\ s':\Omega^{-1}\hookrightarrow \mathcal{A}\otimes\mathrm{Sym}^2 L_2^*\hookrightarrow \mathcal{A}'\otimes\mathrm{Sym}^2 L_2^* \end{array}$$

For $k=1,2$ we have the functor $\mathrm{H}^{\mathcal{Z}_k}:\mathrm{D}(\mathcal{Z}_k)\to\mathrm{D}(X\times\mathcal{Z}_k)$ given by

$$\mathrm{H}^{\mathcal{Z}_k}(K)=(\mathrm{supp}\times\mathfrak{p}_k)_!\mathfrak{q}_k^*K\otimes\bar{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes\langle\gamma,2\check{\rho}\rangle}$$

Similarly, one defines the functor $\mathrm{H}^{\mathcal{Z}_{2,ex}}:\mathrm{D}(\mathcal{Z}_{2,ex})\to\mathrm{D}(X\times\mathcal{Z}_{2,ex})$.

The projection $\alpha_{\mathcal{Z}}:\mathcal{Z}_1\to\mathrm{Bun}_G$ fits into the diagram

$$\begin{array}{ccccc} \mathcal{Z}_1 & \stackrel{\mathfrak{p}_1}{\leftarrow} & \mathcal{Z}_1\times_{\mathrm{Bun}_G}\mathcal{H}_G & \stackrel{\mathfrak{q}_1}{\to} & \mathcal{Z}_1 \\ \downarrow\alpha_{\mathcal{Z}} & & \downarrow & & \downarrow\alpha_{\mathcal{Z}} \\ \mathrm{Bun}_G & \stackrel{\mathfrak{p}}{\leftarrow} & \mathcal{H}_G & \stackrel{\mathfrak{q}}{\to} & \mathrm{Bun}_G \end{array}$$

So, $(\mathrm{id}\times\alpha_{\mathcal{Z}})^*\circ\mathrm{H}\widetilde{\to}\mathrm{H}^{\mathcal{Z}_1}\circ\alpha_{\mathcal{Z}}^*[1](\frac{1}{2})$ naturally.

In this normalization the Hecke property on $\mathcal{Z}_k$ (for $k=1,2$) with respect to $\mathrm{H}^{\mathcal{Z}_k}$ and a given local system W on X writes

$$\mathrm{H}^{\mathcal{Z}_k}(K)\widetilde{\to}W\boxtimes K[3](\frac{3}{2})$$

7.2 One defines the category $\mathrm{D}^W(\mathcal{Z}_{2,ex})$ as in Sect. 2.10-2.11. Let us just indicate its description on strata (they are equivariant under the corresponding groupoids).

For $d\ge 0$ let ${}^d\mathcal{Z}_1\subset\mathcal{Z}_1$ be the locally closed substack given by: there is a subbundle $L'_2\subset M$ such that $L_2\subset L'_2$ is a subsheaf with $d=\deg(L'_2/L_2)$. The stack ${}^d\mathcal{Z}_1$ classifies collections: a modification of rank 2 bundles $L_2\subset L'_2$ on X, and an exact sequence $0\to\mathrm{Sym}^2 L'_2\to ?\to\mathcal{A}\to 0$, where $\mathcal{A}$ is a line bundle on X.

Let ${}^d\mathcal{Z}_{2,ex}=\mathcal{Z}_{2,ex}\times_{\mathcal{Z}_1}{}^d\mathcal{Z}_1$. An analog of Lemma 1 holds for this stratification of $\mathcal{Z}_{2,ex}$, so it suffices to describe the categories $\mathrm{D}^W({}^d\mathcal{Z}_{2,ex})$ for each d.

Let ${}^d\mathcal{Z}'_{2,ex}\hookrightarrow{}^d\mathcal{Z}_{2,ex}$ be the closed substack given by the condition: s factors as

$$\Omega^{-1}\to\mathcal{A}\otimes\mathrm{Sym}^2 L_2'^*\hookrightarrow\mathcal{A}\otimes\mathrm{Sym}^2 L_2^*$$

Let ${}^d\chi_{2,ex}:{}^d\mathcal{Z}'_{2,ex}\to\mathbb{A}^1$ be the map that pairs s with the extension $0\to\mathrm{Sym}^2 L'_2\to ?\to\mathcal{A}\to 0$.

Let ${}^d\mathcal{P}_{2,ex}$ be the stack classifying: a modification of rank 2 bundles $L_2 \subset L_2'$ on X with $d = \deg(L_2'/L_2)$, a line bundle $\mathcal{A}$ on X and a section $s : \mathrm{Sym}^2 L_2' \to \Omega \otimes \mathcal{A}$. Let

$$\phi_{2,ex} : {}^d\mathcal{Z}_{2,ex}' \to {}^d\mathcal{P}_{2,ex}$$

be the projection.

Lemma 6. *Any object of* $\mathrm{D}^W({}^d\mathcal{Z}_{2,ex})$ *is supported at* ${}^d\mathcal{Z}_{2,ex}'$*. The functor*

$${}^dJ(K) = {}^d\chi_{2,ex}^*\mathcal{L}_\psi \otimes \phi_{2,ex}^* K[1](\frac{1}{2})^{\otimes \dim.\mathrm{rel}}$$

provides an equivalence of categories ${}^dJ : \mathrm{D}({}^d\mathcal{P}_{2,ex}) \widetilde{\to} \mathrm{D}^W({}^d\mathcal{Z}_{2,ex})$*. Here* $\dim.\mathrm{rel}$ *is a function of a connected component of* ${}^d\mathcal{Z}_{2,ex}'$ *given by* $\dim.\mathrm{rel} = -\chi(\mathcal{A}^{-1} \otimes \mathrm{Sym}^2 L_2')$*.* $\square$

One mimics the proof of Theorem 1 to get

Theorem 3. *There is an equivalence of categories* $WZ_{1,2,ex} : \mathrm{D}(\mathcal{Z}_1) \widetilde{\to} \mathrm{D}^W(\mathcal{Z}_{2,ex})$*, which is t-exact, and* $(\pi_{2,1,ex})_!$ *is quasi-inverse to it. Moreover, for any* $K \in \mathrm{D}^W(\mathcal{Z}_{2,ex})$ *the natural map* $(\pi_{2,1,ex})_! K \to (\pi_{2,1,ex})_* K$ *is an isomorphism.* $\square$

Let us just explain what this functor does on strata. We have the functor

$${}^dWZ_{1,2,ex} : \mathrm{D}({}^d\mathcal{Z}_1) \to \mathrm{D}^W({}^d\mathcal{Z}_{2,ex})$$

defined as the composition

$$\mathrm{D}({}^d\mathcal{Z}_1) \stackrel{\mathrm{Four}}{\to} \mathrm{D}({}^d\mathcal{P}_{2,ex}) \stackrel{{}^dJ}{\to} \mathrm{D}^W({}^d\mathcal{Z}_{2,ex})$$

If $K \in \mathrm{D}(\mathcal{Z}_1)$ is the extension by zero from ${}^d\mathcal{Z}_1$ then $WZ_{1,2,ex}(K)$ is the extension by zero of ${}^dWZ_{1,2,ex}(K)$ under ${}^d\mathcal{Z}_{2,ex} \hookrightarrow \mathcal{Z}_{2,ex}$.

7.3 Relation to hyper-cuspidality Denote by $\mathcal{Z}_1^{P_0}$ the stack of collections: P_0-torsor on X, that is, an exact sequence $0 \to L_1 \to L_{-1} \to L_{-1}/L_1 \to 0$ on X with $L_1 \in \mathrm{Bun}_1$, $L_{-1} \in \mathrm{Bun}_3$; and an isotropic subsheaf $L_2 \subset L_{-1}$ with $L_2 \in \mathrm{Bun}_2$. Here 'isotropic' means that the composition $\wedge^2 L_2 \to \wedge^2 L_{-1} \to \wedge^2(L_{-1}/L_1)$ vanishes.

Denote by

$$\pi_{2,1,ex}^{P_0} : \mathcal{Z}_{2,ex}^{P_0} \to \mathcal{Z}_1^{P_0}$$

the stack over $\mathcal{Z}_1^{P_0}$ with fibre consisting of all maps (11), where $\mathcal{A} = \det(L_{-1}/L_1)$.

We have a natural diagram

$$\begin{tikzcd}
\mathcal{Z}_{2,ex} \arrow[d] & \mathcal{Z}_{2,ex}^{P_1} \arrow[l, "\beta_{P_1}^{2,ex}"'] \arrow[r, "\alpha_{P_0}^{2,ex}"] \arrow[d] & \mathcal{Z}_{2,ex}^{P_0} \arrow[d, "\pi_{2,1,ex}^{P_0}"] \\
\mathcal{Z}_1 \arrow[d] & \mathcal{Z}_1^{P_1} \arrow[l, "\beta_{P_1}^1"'] \arrow[r, "\alpha_{P_0}^1"] \arrow[d] & \mathcal{Z}_1^{P_0} \arrow[d] \\
\mathrm{Bun}_G & \mathrm{Bun}_{P_1} \arrow[l, "\beta_{P_1}"'] \arrow[r, "\alpha_{P_0}"] & \mathrm{Bun}_{P_0},
\end{tikzcd}$$

where both right squares are cartesian (thus defining the stacks in the middle column).

The constant term functor

$$\mathrm{CT}^{\mathcal{Z}_1}_{P_0} : \mathrm{D}(\mathcal{Z}_1) \to \mathrm{D}(\mathcal{Z}_1^{P_0})$$

is defined by $\mathrm{CT}^{\mathcal{Z}_1}_{P_0}(K) = (\alpha^{\mathcal{Z}}_{P_0})_!(\beta^{\mathcal{Z}}_{P_1})^* K$. Similarly, $\mathrm{CT}^{\mathcal{Z}_{2,ex}}_{P_0} : \mathrm{D}(\mathcal{Z}_{2,ex}) \to \mathrm{D}(\mathcal{Z}^{P_0}_{2,ex})$ is defined as

$$\mathrm{CT}^{\mathcal{Z}_{2,ex}}_{P_0}(K) = (\alpha^{2,ex}_{P_0})_!(\beta^{2,ex}_{P_1})^* K$$

Definition 3. A complex $K \in \mathrm{D}(\mathcal{Z}_1)$ (resp., $K \in \mathrm{D}^W(\mathcal{Z}_{2,ex})$) is *hyper-cuspidal* if $\mathrm{CT}^{\mathcal{Z}_1}_{P_0}(K) = 0$ (resp., $\mathrm{CT}^{\mathcal{Z}_{2,ex}}_{P_0}(K) = 0$). We denote by $\mathrm{D}_{hcusp}(\mathcal{Z}_1) \subset \mathrm{D}(\mathcal{Z}_1)$ and $\mathrm{D}^W_{hcusp}(\mathcal{Z}_2) \subset \mathrm{D}^W(\mathcal{Z}_{2,ex})$ the full triangulated subcategories of hyper-cuspidal objects.

Clearly, $K \in \mathrm{D}(\mathrm{Bun}_G)$ is hyper-cuspidal iff $\alpha^*_{\mathcal{Z}} K \in \mathrm{D}(\mathcal{Z}_1)$ is hyper-cuspidal. The following is easy to prove.

Proposition 8. *1) A complex $K \in \mathrm{D}^W(\mathcal{Z}_{2,ex})$ is hyper-cuspidal if and only if the following holds: for any k-point $z = (L_2 \subset M,\ s : \mathrm{Sym}^2 L_2 \to \mathcal{A} \otimes \Omega)$ such that L_2 has a rank 1 isotropic subbundle (with respect to the form s) we have $K_z = 0$.*
2) The functor $WZ_{1,2,ex} : \mathrm{D}(\mathcal{Z}_1) \widetilde{\to} \mathrm{D}^W(\mathcal{Z}_{2,ex})$ induces an equivalence of triangulated categories

$$\mathrm{D}_{hcusp}(\mathcal{Z}_1) \widetilde{\to} \mathrm{D}^W_{hcusp}(\mathcal{Z}_2) \qquad \square$$

Remarks . i) For each integer d we have a closed substack $Y_d \hookrightarrow \mathcal{Z}_{2,ex}$ given by the condition that L_2 admits an isotropic rank 1 subbundle (with respect to s) of degree $\geq d$. We have $Y_d \subset Y_{d-1} \subset \dots$. A complex $K \in \mathrm{D}^W(\mathcal{Z}_{2,ex})$ is hyper-cuspidal if and only if its $*$-restriction to each Y_d vanishes.
ii) If $s : \mathrm{Sym}^2 L_2 \to \mathcal{A} \otimes \Omega$ is such that L_2 has no rank 1 isotropic subbundles then the form s is generically nondegenerate, that is, $L_2 \hookrightarrow L_2^* \otimes \mathcal{A} \otimes \Omega$ is an inclusion.

Hecke functors preserve our equivariance conditions as well as hyper-cuspidality. Moreover, they commute with $WZ_{1,2,ex}$, namely as in Sect. 5 one proves

Proposition 9. *1) The functor $\mathrm{H}^{\mathcal{Z}_{2,ex}}$ sends $\mathrm{D}^W(\mathcal{Z}_{2,ex})$ to $\mathrm{D}^W(X \times \mathcal{Z}_{2,ex})$ and $\mathrm{D}^W_{hcusp}(\mathcal{Z}_2)$ to $\mathrm{D}^W_{hcusp}(X \times \mathcal{Z}_2)$.*
2) The functor $\mathrm{H}^{\mathcal{Z}_1}$ sends $\mathrm{D}_{hcusp}(\mathcal{Z}_1)$ to $\mathrm{D}_{hcusp}(X \times \mathcal{Z}_1)$.
3) We have a canonical isomorphism of functors $\mathrm{H}^{\mathcal{Z}_{2,ex}} \circ WZ_{1,2,ex} \widetilde{\to} (\mathrm{id} \times WZ_{1,2,ex}) \circ \mathrm{H}^{\mathcal{Z}_1}$ from $\mathrm{D}(\mathcal{Z}_1)$ to $\mathrm{D}^W(X \times \mathcal{Z}_{2,ex})$. $\square$

7.4 In this subsection we prove the following generalization of ([5], Theorem 7.9).

Proposition 10. *The functor $\mathrm{H}^{\mathcal{Z}_2} : \mathrm{D}(\mathcal{Z}_2) \to \mathrm{D}(X \times \mathcal{Z}_2)$ is right-exact for the perverse t-structures.*

Let $\pi_{3,2} : \mathcal{Z}_3 \to \mathcal{Z}_2$ denote the stack classifying $(L_2 \subset M,\ s : \Omega^{-1} \hookrightarrow \mathcal{A} \otimes \mathrm{Sym}^2 L_2^*) \in \mathcal{Z}_2$ together with a line subbundle $L_1 \subset L_2$ such that

$$\mathrm{H}^1(X, L_1^{-1} \otimes (L_2/L_1)) = 0$$

The projection $\pi_{3,2}$ is smooth and surjective. Consider the diagram

$$\begin{array}{ccccc} \mathcal{Z}_3 & \stackrel{\mathfrak{p}_3}{\leftarrow} & \mathcal{Z}_3 \times_{\mathrm{Bun}_G} \mathcal{H}_G & \stackrel{\mathfrak{q}_3}{\to} & \mathcal{Z}_3 \\ \downarrow \pi_{3,2} & & \downarrow & & \downarrow \pi_{3,2} \\ \mathcal{Z}_2 & \stackrel{\mathfrak{p}_2}{\leftarrow} & \mathcal{Z}_2 \times_{\mathrm{Bun}_G} \mathcal{H}_G & \stackrel{\mathfrak{q}_2}{\to} & \mathcal{Z}_2, \end{array}$$

where the left square is cartesian. Define $\mathrm{H}^{\mathcal{Z}_3} : \mathrm{D}(\mathcal{Z}_3) \to \mathrm{D}(X \times \mathcal{Z}_3)$ by

$$\mathrm{H}^{\mathcal{Z}_3}(K) = (\mathrm{supp} \times \mathfrak{p}_3)_! \mathfrak{q}_3^* K \otimes \bar{\mathbb{Q}}_\ell(\frac{1}{2})[1]^{\otimes \langle \gamma, 2\check{\rho} \rangle}$$

For $K \in \mathrm{D}(\mathcal{Z}_2)$ we have $\pi_{3,2}^* \mathrm{H}^{\mathcal{Z}_2}(K)[\dim] \widetilde{\to} \mathrm{H}^{\mathcal{Z}_3}(\pi_{3,2}^* K)[\dim]$, where dim is a funtion of a connected component of $\mathcal{Z}_3$, namely the relative dimension of the corresponding component over $\mathcal{Z}_2$. Since $\pi_{3,2}^*[\dim]$ is exact, it suffices to show that $\mathrm{H}^{\mathcal{Z}_3}$ is right-exact.

For $\bar{d} = (d_1, d_2)$ with $0 \le d_1 \le d_2$ denote by ${}^{\bar{d}}\mathcal{Z}_3 \subset \mathcal{Z}_3$ the locally closed substack given by the condition that there exist a diagram

$$\begin{array}{ccccc} \bar{L}_1 & \subset & \bar{L}_2 & \subset & M \\ \cup & & \cup & & \\ L_1 & \subset & L_2, & & \end{array}$$

where $\bar{L}_k \subset M$ is a subbundle of rank k with $\deg(\bar{L}_k / L_k) = d_k$. The stacks ${}^{\bar{d}}\mathcal{Z}_3$ form a stratification of $\mathcal{Z}_3$.

For $x \in X$ let ${}_x\mathcal{H}_G \subset \mathcal{H}_G$ denote the preimage of x under $\mathrm{supp} : \mathcal{H}_G \to X$. The following is straightforward.

Lemma 7. *For a k-point of $\mathcal{Z}_3$ let D be the effective divisor such that $s : \Omega^{-1}(D) \hookrightarrow \mathcal{A} \otimes \mathrm{Sym}^2 L_2^*$ is a subbundle. Then the fibre of $\mathfrak{q}_3 : \mathcal{Z}_3 \times_{\mathrm{Bun}_G} \mathcal{H}_G \to \mathcal{Z}_3$ over this point is contained in*

$$\bigcup_{x \in \mathrm{supp}(D)} \mathcal{Z}_3 \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G$$

□

Given $\bar{d} = (d_1, d_2)$ and $\bar{d}' = (d_1', d_2')$ denote by ${}^{\bar{d}, \bar{d}'}\mathcal{Z}_3 \subset \mathcal{Z}_3 \times_{\mathrm{Bun}_G} \mathcal{H}_G$ the intersection

$$(\mathfrak{p}_3)^{-1}({}^{\bar{d}}\mathcal{Z}_3) \cap (\mathfrak{q}_3)^{-1}({}^{\bar{d}'}\mathcal{Z}_3)$$

For $x \in X$ let ${}^{\bar{d}, \bar{d}'}_{x}\mathcal{Z}_3$ denote the intersection of ${}^{\bar{d}, \bar{d}'}\mathcal{Z}_3$ with $\mathcal{Z}_3 \times_{\mathrm{Bun}_G} {}_x\mathcal{H}_G$. Combining Lemma 7 with ([5], Lemma 7.11), we are reduced to the following statement.

Lemma 8. *For any $\bar{d}, \bar{d}'$ and $x \in X$ the sum of (the maximum of) the dimensions of fibres of maps in the diagram*

$$\mathcal{Z}_3 \overset{\mathfrak{p}_3}{\leftarrow} {}_x^{\bar{d},\bar{d}'}\mathcal{Z}_3 \overset{\mathfrak{q}_3}{\to} \mathcal{Z}_3 \tag{12}$$

does not exceed $\langle \gamma, 2\check{\rho} \rangle = 3$.

Proof A point of ${}_x^{\bar{d},\bar{d}'}\mathcal{Z}_3$ gives rise to the diagram

$$\begin{array}{ccccc} \bar{L}'_1 & \subset & \bar{L}'_2 & \subset & M' \\ \cup & & \cup & & \cup \\ \bar{L}_1 & \subset & \bar{L}_2 & \subset & M \\ \cup & & \cup & & \\ L_1 & \subset & L_2, & & \end{array}$$

with $d_k = \deg(\bar{L}_k/L_k)$ and $d'_k = \deg(\bar{L}'_k/L_k)$. We must examine the cases:

1) $\bar{d} = \bar{d}'$. In this case a fibre of $\mathfrak{q}_3$ is a point, because $\bar{L}'_2$ generates a lagrangian subspace in $M'/M'(-x)$. A fibre of $\mathfrak{p}_3$ is 3-dimensional.

2) $d'_1 = d_1$, $d'_2 = d_2 + 1$. Then a fibre of $\mathfrak{q}_3$ is 1-dimensional, becuase M must contain $\bar{L}'_1$. A fibre of $\mathfrak{p}_3$ is 2-dimensional.

3) $d'_1 = d_1 + 1$, $d'_2 = d_2 + 1$. Then $\bar{L}'_2 = \bar{L}_2 + \bar{L}'_1$ and $\bar{L}'_1 = \bar{L}_1(x)$. A fibre of $\mathfrak{q}_3$ is 2-dimensional, a fibre of $\mathfrak{p}_3$ is 1-dimensional.

4) $d'_1 = d_1 + 1$, $d'_2 = d_2 + 2$. Then $\bar{L}'_2 = \bar{L}_2(x)$. A fibre of $\mathfrak{p}_3$ is a point, because $M' = M + \bar{L}'_2$. A fibre of $\mathfrak{q}_3$ is 3-dimenisonal. $\square$

7.5 Hecke functors on $\mathcal{P}_2$

Recall the stack ${}^d\mathcal{P}_2$ classifying collections: a modification of rank 2 bundles $(L_2 \subset L'_2)$ on X with $d = \deg(L'_2/L_2)$, $\mathcal{A} \in \mathrm{Bun}_1$, and a section $s : \Omega^{-1} \hookrightarrow \mathcal{A} \otimes \mathrm{Sym}^2 L'^*_2$. Lemma 6 yields the equivalence of categories ${}^dJ : \mathrm{D}({}^d\mathcal{P}_2) \widetilde{\to} \mathrm{D}^W({}^d\mathcal{Z}_2)$.

We are going to define for $i = 0, 1, 2$ the functors

$${}_i\mathrm{H}^{\mathcal{P}} : \mathrm{D}({}^{d+i}\mathcal{P}_2) \to \mathrm{D}(X \times {}^d\mathcal{P}_2)$$

which, by construction, will satisfy the following property.

Proposition 11. *Let $K \in \mathrm{D}^W(\mathcal{Z}_2)$, ${}^dK \in \mathrm{D}({}^d\mathcal{P}_2)$ and ${}^dF \in \mathrm{D}(X \times {}^d\mathcal{P}_2)$. Assume given for each d isomorphisms*

$${}^dJ({}^dK) \widetilde{\to} K \mid_{{}^d\mathcal{Z}'_2} \quad \textit{and} \quad {}^dJ({}^dF) \widetilde{\to} \mathrm{H}^{\mathcal{Z}_2}(K) \mid_{X \times {}^d\mathcal{Z}'_2},$$

where we used the $*$-restrictions. Then ${}^dF$ is an extension of objects ${}_i\mathrm{H}^{\mathcal{P}}({}^{d+i}K)$ $(i = 0, 1, 2)$ in the triangulated category $\mathrm{D}(X \times {}^d\mathcal{P}_2)$. More precisely, there exist distinguished triangles in $\mathrm{D}(X \times {}^d\mathcal{P}_2)$

$$C \to {}^dF \to {}_2\mathrm{H}^{\mathcal{P}}({}^{d+2}K) \quad \textit{and} \quad {}_0\mathrm{H}^{\mathcal{P}}({}^dK) \to C \to {}_1\mathrm{H}^{\mathcal{P}}({}^{d+1}K)$$

7.5.1 Let $\delta_0 : X\times {}^d\mathcal{P}_2\to {}^d\mathcal{P}_2$ be the map sending $(x\in X, L_2\subset L_2', \mathcal{A}, s)$ to $(L_2\subset L_2', \mathcal{A}(x), s')$, where s' is the composition

$$\operatorname{Sym}^2 L_2' \xrightarrow{s} \mathcal{A}\otimes\Omega\hookrightarrow \mathcal{A}(x)\otimes\Omega$$

Set ${}_0\mathrm{H}^{\mathcal{P}}(S)=\delta_0^*S$. Since δ_0 is quasi-finite, ${}_0\mathrm{H}^{\mathcal{P}}$ is right exact for the perverse t-structures. Consider the diagram

$$X\times {}^d\mathcal{P}_2 \stackrel{\operatorname{id}\times\delta_0}{\leftarrow} X\times {}^d\mathcal{P}_2 \stackrel{\delta_2}{\to} {}^{d+2}\mathcal{P}_2,$$

where δ_2 sends $(x\in X, L_2\subset L_2', \mathcal{A}, s)$ to $(L_2\subset L_2'(x), \mathcal{A}(2x), s)$. Note that $\operatorname{id}\times\delta_0$ is a closed immersion. Set

$${}_2\mathrm{H}^{\mathcal{P}}(S)=(\operatorname{id}\times\delta_0)_!\delta_2^*S$$

Since δ_2 is quasi-finite, ${}_2\mathrm{H}^{\mathcal{P}}$ is right exact for the perverse t-structures.

Let ${}^d\mathcal{H}_{\mathcal{P}}$ denote the stack of collections: $\mathcal{A}\in\operatorname{Bun}_1$, modifications of rank 2 vector bundles $L_2\subset L_2'\subset L_2''$ with $d=\deg(L_2'/L_2)$, where L_2''/L_2' is a torsion sheaf of length one supported at $x\in X$, and a commutative diagram

$$\begin{array}{ccc} \operatorname{Sym}^2 L_2'' & \to & \mathcal{A}\otimes\Omega(x)\\ \cup & & \cup\\ \operatorname{Sym}^2 L_2' & \stackrel{s}{\to} & \mathcal{A}\otimes\Omega \end{array} \tag{13}$$

with $s\neq 0$.

The existence of the latter diagram means that L_2''/L_2' is an isotropic subspace of $L_2'(x)/L_2'$ equiped with the form $s:\operatorname{Sym}^2(L_2'(x)/L_2')\to(\mathcal{A}(2x)/\mathcal{A}(x))\otimes\Omega$.

We have the diagram

$$X\times {}^d\mathcal{P}_2 \stackrel{\operatorname{supp}\times\mathfrak{p}_{\mathcal{P}}}{\leftarrow} {}^d\mathcal{H}_{\mathcal{P}} \stackrel{\mathfrak{q}_{\mathcal{P}}}{\to} {}^{d+1}\mathcal{P}_2,$$

where $\mathfrak{p}_{\mathcal{P}}$ sends a point of ${}^d\mathcal{H}_{\mathcal{P}}$ to $(L_2\subset L_2', \mathcal{A}, s)$. The map $\mathfrak{q}_{\mathcal{P}}$ sends a point of ${}^d\mathcal{H}_{\mathcal{P}}$ to

$$(L_2\subset L_2'', \mathcal{A}(x), s)$$

The map $\operatorname{supp}: {}^d\mathcal{H}_{\mathcal{P}}\to X$ sends a point of ${}^d\mathcal{H}_{\mathcal{P}}$ as above to x. We set

$${}_1\mathrm{H}^{\mathcal{P}}(S)=(\operatorname{supp}\times\mathfrak{p}_{\mathcal{P}})_!\mathfrak{q}_{\mathcal{P}}^*S[1](\frac{1}{2})$$

Proof of Proposition 11

Recall the diagram we used to define the functor $\mathrm{H}^{\mathcal{Z}_2}$

$$X\times\mathcal{Z}_2 \stackrel{\operatorname{supp}\times\mathfrak{p}_2}{\leftarrow} \mathcal{Z}_2\times_{\operatorname{Bun}_G}\mathcal{H}_G \stackrel{\mathfrak{q}_2}{\to} \mathcal{Z}_2$$

Given $d\le d'$ set ${}^{d,d'}\mathcal{Z}_2=\mathfrak{q}_2^{-1}({}^{d'}\mathcal{Z}_2')\cap\mathfrak{p}_2^{-1}({}^d\mathcal{Z}_2')$. We will calculate the direct image under $\operatorname{supp}\times\mathfrak{p}_2$ with respect to the corresponding stratification of $\mathcal{Z}_2\times_{\operatorname{Bun}_G}\mathcal{H}_G$. Let u,v denote the maps in the induced diagram

$$X\times {}^d\mathcal{Z}_2' \stackrel{u}{\leftarrow} {}^{d,d'}\mathcal{Z}_2 \stackrel{v}{\to} {}^{d'}\mathcal{Z}_2'$$

Let $\tilde{K}$ be the $*$-restriction of $\mathfrak{q}_2^* K$ to ${}^{d,d'}\mathcal{Z}_2$. A point of ${}^{d,d'}\mathcal{Z}_2$ gives rise to the diagram

$$\begin{array}{ccc} L_2'' & \subset & M' \\ \cup & & \cup \\ L_2' & \subset & M \\ \cup & & \\ L_2 & & \end{array}$$

with $d = \deg(L_2'/L_2)$ and $d' = \deg(L_2''/L_2)$. We must examine three cases:

1) $d = d'$. Then $L_2'' = L_2'$, and a fibre of u admits a free transitive action of the geometric fibre $(\mathcal{A}^{-1} \otimes \operatorname{Sym}^2 L_2')_x$. The complex $\tilde{K}$ is constant along the fibres of u. So, ${}_0\mathrm{H}^{\mathcal{P}}({}^d K)$ is the contribution in ${}^d F$ of the stratum ${}^{d,d}\mathcal{Z}_2$

2) $d' = d+2$. For a k-point of $X \times {}^d\mathcal{Z}_2'$ we have $L_2'' = L_2(x)$ and $M' = L_2'' + M$ in the above diagram. If $\operatorname{Sym}^2 L_2' \to \mathcal{A} \otimes \Omega$ does not factor through $\mathcal{A} \otimes \Omega(-x)$ then the fibre of u over this point is empty, otherwise this fibre is a point scheme. In the second case the extension $0 \to \operatorname{Sym}^2 L_2'' \to ? \to \mathcal{A}(x) \to 0$ is the push-forward of $0 \to (\operatorname{Sym}^2 L_2')(x) \to ? \to \mathcal{A}(x) \to 0$ under

$$(\operatorname{Sym}^2 L_2')(x) \hookrightarrow (\operatorname{Sym}^2 L_2')(2x)$$

So, the contribution of ${}^{d,d+2}\mathcal{Z}_2$ in ${}^d F$ is ${}_2\mathrm{H}^{\mathcal{P}}({}^{d+2}K)$.

3) $d' = d+1$. Fix a k-point of $X \times {}^d\mathcal{Z}_2'$ and denote by $\bar{Y}$ the corresponding fibre of u.

Let Y be the scheme of L_2'' such that $L_2' \subset L_2'' \subset L_2'(x)$ gives rise to the diagram (13). Note that $Y \widetilde{\to} \mathbb{P}^1$ if the form on $L_2'(x)/L_2'$ is zero, Y is a point if the kernel of the corresponding form is 1-dimensional, and Y consists of two points if the form on $L_2'(x)/L_2'$ is non degenerate.

The fibres of the projection $\bar{Y} \to Y$ are isomorphic to $\mathbb{A}^1$. More precisely, the 1-dimensional space $\mathcal{A}_x^{-1} \otimes \operatorname{Sym}^2(L_2'/L_2''(-x))$ acts on a fibre freely and transitively.

To see that the restriction $\tilde{K} \mid_{\bar{Y}}$ is constant along the fibres of $\bar{Y} \to Y$, note that the morphism $\mathcal{A}^{-1} \otimes (\operatorname{Sym}^2 L_2'')(-x) \to \Omega$ factors as

$$\mathcal{A}^{-1} \otimes (\operatorname{Sym}^2 L_2'')(-x) \hookrightarrow \mathcal{N} \to \Omega,$$

where $\mathcal{N}$ is the upper modification of $\mathcal{A}^{-1} \otimes (\operatorname{Sym}^2 L_2'')(-x)$ defined by the 1-dimensional subspace $\mathcal{A}_x^{-1} \otimes \operatorname{Sym}^2(L_2'/L_2''(-x))$ in the geometric fibre $\mathcal{A}^{-1} \otimes \operatorname{Sym}^2(L_2'')_x$.

It easily follows that the contribution of ${}^{d,d+1}\mathcal{Z}_2$ in ${}^d F$ is ${}_1\mathrm{H}^{\mathcal{P}}({}^{d+1}K)$. $\square$

Corolary 2. *Let $K \in \mathrm{D}^W(\mathcal{Z}_2)$ and ${}^d K \in \mathrm{D}({}^d\mathcal{P}_2)$ equiped with isomorphisms ${}^d J({}^d K) \widetilde{\to} K \mid_{{}^d\mathcal{Z}_2'}$. Assume*

$$\mathrm{H}^{\mathcal{Z}_2}(K) \widetilde{\to} W \boxtimes K[3](\frac{3}{2})$$

for a local system W on X. Then for each d the complex $W \boxtimes {}^d K[3](\frac{3}{2})$ is an extension of objects ${}_i\mathrm{H}^{\mathcal{P}}({}^{d+i}K)$ $(i = 0, 1, 2)$ in the triangulated category $\mathrm{D}(X \times {}^d\mathcal{P}_2)$. $\square$

7.6 Hecke functors on $\bar{\mathcal{S}}$

Let $\bar{\mathcal{S}}$ denote the stack classifying $L_2 \in \mathrm{Bun}_2$, $\mathcal{A} \in \mathrm{Bun}_1$ and an inclusion of coherent sheaves $s : \Omega^{-1} \hookrightarrow \mathcal{A} \otimes \mathrm{Sym}^2 L_2^*$. Define the following Hecke operators for $i = 0, 1, 2$

$$ {}_i\mathrm{H}^{\bar{\mathcal{S}}} : \mathrm{D}(\bar{\mathcal{S}}) \to \mathrm{D}(X \times \bar{\mathcal{S}}) $$

Let $\delta_0 : X \times \bar{\mathcal{S}} \to \bar{\mathcal{S}}$ be the map sending $(x \in X, L_2, \mathcal{A}, s)$ to $(L_2, \mathcal{A}(x), s')$, where s' is the composition

$$ \mathrm{Sym}^2 L_2 \xrightarrow{s} \mathcal{A} \otimes \Omega \hookrightarrow \mathcal{A}(x) \otimes \Omega $$

Set ${}_0\mathrm{H}^{\bar{\mathcal{S}}}(K) = \delta_0^* K$. Since δ_0 is quasi-finite, ${}_0\mathrm{H}^{\bar{\mathcal{S}}}$ is right exact for the perverse t-structures.

Consider the diagram

$$ X \times \bar{\mathcal{S}} \stackrel{\mathrm{id} \times \delta_0}{\longleftarrow} X \times \bar{\mathcal{S}} \stackrel{\delta_2}{\to} \bar{\mathcal{S}}, $$

where δ_2 sends $(x \in X, L_2, \mathcal{A}, s)$ to $(L_2(x), \mathcal{A}(2x), s)$. Note that $\mathrm{id} \times \delta_0$ is a closed immersion. Set

$$ {}_2\mathrm{H}^{\bar{\mathcal{S}}}(K) = (\mathrm{id} \times \delta_0)_! \delta_2^* K $$

Let $\mathcal{H}_{\bar{\mathcal{S}}}$ denote the stack of collections: $\mathcal{A} \in \mathrm{Bun}_1$, modifications of rank 2 vector bundles $L_2 \subset L_2'$, with $\mathrm{div}(L_2'/L_2) = x$, and a commutative diagram

$$ \begin{array}{ccc} \mathrm{Sym}^2 L_2' & \to & \mathcal{A} \otimes \Omega(x) \\ \cup & & \cup \\ \mathrm{Sym}^2 L_2 & \stackrel{s}{\to} & \mathcal{A} \otimes \Omega \end{array} \tag{14} $$

with $s \neq 0$. The existence of the latter diagram means that L_2'/L_2 is an isotropic subspace of $L_2(x)/L_2$ equiped with the form $s : \mathrm{Sym}^2(L_2(x)/L_2) \to (\mathcal{A}(2x)/\mathcal{A}(x)) \otimes \Omega$.

We have the diagram

$$ X \times \bar{\mathcal{S}} \stackrel{\mathrm{supp} \times \mathfrak{p}_{\bar{\mathcal{S}}}}{\longleftarrow} \mathcal{H}_{\bar{\mathcal{S}}} \stackrel{\mathfrak{q}_{\bar{\mathcal{S}}}}{\to} \bar{\mathcal{S}}, $$

where $\mathfrak{p}_{\bar{\mathcal{S}}}$ sends a point of $\mathcal{H}_{\bar{\mathcal{S}}}$ to $(L_2, \mathcal{A}, s)$. The map $\mathfrak{q}_{\bar{\mathcal{S}}}$ sends a point of $\mathcal{H}_{\bar{\mathcal{S}}}$ to $(L_2', \mathcal{A}(x), s)$. The map $\mathrm{supp} : \mathcal{H}_{\bar{\mathcal{S}}} \to X$ sends a point (14) to x. Set

$$ {}_1\mathrm{H}^{\bar{\mathcal{S}}}(K) = (\mathrm{supp} \times \mathfrak{p}_{\bar{\mathcal{S}}})_! \mathfrak{q}_{\bar{\mathcal{S}}}^* K[1](\frac{1}{2}) $$

7.6.1 Define the functor $F_{\bar{\mathcal{S}}} : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\bar{\mathcal{S}})$ as follows. Given $K \in \mathrm{D}(\mathrm{Bun}_G)$ set

$$ K_1 = \alpha_{\mathcal{Z}}^* K[\dim.\mathrm{rel}](\frac{\dim.\mathrm{rel}}{2}), \tag{15} $$

where $\dim.\mathrm{rel}$ is the relative dimension of the corresponding connected component of $\mathcal{Z}_1$ over Bun_G. Let K_P denote the restriction of K_1 to the open substack $\mathrm{Bun}_P \subset \mathcal{Z}_1$. Set

$$ F_{\bar{\mathcal{S}}}(K) = \mathrm{Four}(K_P) \mid_{\bar{\mathcal{S}}} $$

Proposition 12. *Let $K \in \mathrm{D}(\mathrm{Bun}_G)$ be a Hecke eigen-sheaf corresponding to a $\check{G}$-local system $W_{\check{G}}$ on X. Set $F = F_{\bar{\mathcal{S}}}(K)$. Consider the local systems $W = W_{\check{G}}^{\check{\omega}_1}$ and $W^0 = W_{\check{G}}^{\check{\omega}_0}$. Then*
1) there exist distinguished triangles in $\mathrm{D}(X \times \bar{\mathcal{S}})$

$$C \to W \boxtimes F \to {}_2\mathrm{H}^{\bar{\mathcal{S}}}(F)[3](\frac{3}{2}) \quad \textit{and} \quad {}_0\mathrm{H}^{\bar{\mathcal{S}}}(F)[-3](\frac{-3}{2}) \to C \to {}_1\mathrm{H}^{\bar{\mathcal{S}}}(F)$$

2) For $\delta_2 : X \times \bar{\mathcal{S}} \to \bar{\mathcal{S}}$ we have $\delta_2^* F \widetilde{\to} W^0 \boxtimes F$.

Proof 1) Let $K_1 \in \mathrm{D}(\mathcal{Z}_1)$ be given by (15). Recall that a point of ${}^d\mathcal{Z}_1$ is given by

$$(\mathcal{A} \in \mathrm{Bun}_2,\ L_2 \subset L_2',\ 0 \to \mathrm{Sym}^2 L_2' \to ? \to \mathcal{A} \to 0)$$

Let $\tau^P : {}^d\mathcal{Z}_1 \to \mathrm{Bun}_P$ be the map forgetting L_2. Calculation of dimensions shows that for the $*$-restriction

$$K_1 \mid_{{}^d\mathcal{Z}_1} \widetilde{\to} (\tau^P)^* K_P[3d](\frac{3d}{2})$$

canonically. Recall that a point of ${}^d\mathcal{P}_2$ is given by $(\mathcal{A} \in \mathrm{Bun}_1, L_2 \subset L_2', s : \mathrm{Sym}^2 L_2' \to \mathcal{A} \otimes \Omega)$. Let $\tau : {}^d\mathcal{P}_2 \to \bar{\mathcal{S}}$ be the map forgetting L_2.

Set $K_2 = WZ_{1,2}(K_1)$. For each d set ${}^dK_2 = \tau^* F_{\bar{\mathcal{S}}}(K)[3d](\frac{3d}{2})$. Then for the $*$-restriction we have canonically

$$K_2 \mid_{{}^d\mathcal{Z}_2} \widetilde{\to} {}^dJ({}^dK_2)$$

One easily checks that for $i = 0, 1, 2$ we have canonical isomorphisms of functors

$$(\mathrm{id} \times \tau)^* \circ {}_i\mathrm{H}^{\bar{\mathcal{S}}} \widetilde{\to} {}_i\mathrm{H}^{\mathcal{P}} \circ \tau^*$$

from $\mathrm{D}(\bar{\mathcal{S}})$ to $\mathrm{D}(X \times {}^d\mathcal{P}_2)$.

By Corolary 2, for each d the complex $W \boxtimes {}^dK_2[3](\frac{3}{2})$ is an extension of objects ${}_i\mathrm{H}^{\mathcal{P}}({}^{d+i}K_2)$ $(i = 0, 1, 2)$ in $\mathrm{D}(X \times {}^d\mathcal{P}_2)$. Specifying to $d = 0$, one gets the desired assertion. $\square$

7.7 The stacks ${}^{rss}\mathcal{S} \subset \mathcal{S} \subset \bar{\mathcal{S}}$

7.7.1 Let $\mathcal{S}$ denote the stack classifying $L \in \mathrm{Bun}_2$, $\mathcal{C} \in \mathrm{Bun}_1$ and a map $\mathrm{Sym}^2 L \to \mathcal{C}$ inducing an inclusion of coherent sheaves $L \hookrightarrow L^* \otimes \mathcal{C}$. The map $\mathcal{S} \to \bar{\mathcal{S}}$ given by $L_2 = L$, $\mathcal{A} = \mathcal{C} \otimes \Omega^{-1}$ is an open immersion.

Since the open substack $\mathcal{S} \subset \bar{\mathcal{S}}$ is defined by a condition at the generic point of X, the Hecke operators ${}_i\mathrm{H}^{\bar{\mathcal{S}}}$ preserve this open substack, we denote the corresponding functors by

$${}_i\mathrm{H}^{\mathcal{S}} : \mathrm{D}(\mathcal{S}) \to \mathrm{D}(X \times \mathcal{S}) \quad (i = 0, 1, 2)$$

Let $\mathcal{S}_d$ denote the union of those components of $\mathcal{S}$ for which $2 \deg \mathcal{C} - 2 \deg L = d$. Note that $d \geq 0$ is even.

The nonramified two-sheeted Galois coverings $\tilde{X} \to X$ are in bijection with $\mathrm{H}^1_{et}(X, \mathbb{Z}/2\mathbb{Z})$, and also in bijection with the isomorphism classes of pairs $(\mathcal{E}, \kappa)$, where $\mathcal{E}$ is a line bundle on X and $\kappa : \mathcal{E}^{\otimes 2} \widetilde{\to} \mathcal{O}$.

Lemma 9. *The stack $\mathcal{S}_0$ classifies pairs: a two-sheeted nonramified covering $\tilde{X} \to X$ and a line bundle $\mathcal{B}$ on $\tilde{X}$.*

Proof 1) Given a Galois covering $\pi : \tilde{X} \to X$ of degree 2 and a line bundle $\mathcal{B}$ on $\tilde{X}$ set $L = \pi_*\mathcal{B}$. Let σ be the nontrivial automorphism of $\tilde{X}$ over X. Let $\mathcal{E}$ denote the anti-invariants of $\pi_*\mathcal{O}_{\tilde{X}}$ under σ, so $\pi_*\mathcal{O}_{\tilde{X}} \widetilde{\to} \mathcal{O}_X \oplus \mathcal{E}$. Note that $\pi^*\mathcal{E} \widetilde{\to} \mathcal{O}_{\tilde{X}}$ is equiped with the nontrivial descent data

$$\sigma^*\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}} \stackrel{-1}{\to} \mathcal{O}_{\tilde{X}}$$

Set $\mathcal{C} = (\det L) \otimes \mathcal{E}$, so $\pi^*\mathcal{C}$ identifies with $\mathcal{B} \otimes \sigma^*\mathcal{B}$ equiped with the natural descent data

$$\sigma^*(\mathcal{B} \otimes \sigma^*\mathcal{B}) = \mathcal{B} \otimes \sigma^*\mathcal{B} \stackrel{\mathrm{id}}{\to} \mathcal{B} \otimes \sigma^*\mathcal{B}$$

We have canonically $\pi^*L \widetilde{\to} \mathcal{B} \oplus \sigma^*\mathcal{B}$, where σ acts on $\mathcal{B} \oplus \sigma^*\mathcal{B}$ naturally. The projection

$$\mathrm{Sym}^2(\mathcal{B} \oplus \sigma^*\mathcal{B}) \to \mathcal{B} \otimes \sigma^*\mathcal{B}$$

with the natural descent data gives rise to a map $\mathrm{Sym}^2 L \to \mathcal{C}$, which is a point of $\mathcal{S}_0$.

2) On the other side, let $s : \mathrm{Sym}^2 L \to \mathcal{C}$ be a point of $\mathcal{S}_0$. Let $\tilde{X} \subset \mathbb{P}(L)$ be the two-sheeted covering of X whose fibre over $x \in X$ is the set of isotropic subspaces in $(L)_x$. Let $\mathcal{B}$ be the line bundle on $\tilde{X}$ whose fibre at $V \subset L_x$ is V itself. For $\pi : \tilde{X} \to X$ we get $\pi_*\mathcal{B} \widetilde{\to} L$ canonically.

Let σ be the nontrivial automorphism of $\tilde{X}$ over X. Let $\mathcal{E}$ denote the σ-anti-invariants in $\pi_*\mathcal{O}$. By 1), we have the symmetric form $\mathrm{Sym}^2 L \to \mathcal{E} \otimes \det L$, which is also a point of $\mathcal{S}_0$. Let E denote the kernel of $\mathrm{Sym}^2 L \to \mathcal{E} \otimes \det L$. Let us show that the composition

$$E \to \mathrm{Sym}^2 L \to \mathcal{C}$$

vanishes. It suffices to prove this after applying π^*, but $\pi^*E \widetilde{\to} \mathcal{B}^{\otimes 2} \oplus \sigma^*\mathcal{B}^{\otimes 2}$. So, we get a map τ_L included into the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^2 L & \to & \mathcal{E} \otimes \det L \\ \downarrow & \swarrow \tau_L & \\ \mathcal{C} & & \end{array}$$

Since both symmetric forms on L are everywhere nondegenerate, τ_L is an isomorphism. $\square$

Remark 6. i) A version holds for a curve which may be not complete.
ii) If in the above lemma $\mathcal{B} = \mathcal{O}$ on $\tilde{X}$ then $\mathbb{Z}/2\mathbb{Z}$ acts on $\pi_*\mathcal{B} = L$. So, $L \widetilde{\to} \mathcal{O} \oplus \mathcal{E}$, where $\mathcal{E}$ is the line bundle of anti-invariants. The map $\mathrm{Sym}^2 L \to \mathcal{E} \otimes \det L$ becomes $\mathcal{O} \oplus \mathcal{E} \oplus \mathcal{E}^{\otimes 2} \to \mathcal{O}$, it is given by $(1, 0, -\kappa)$. The curve $\tilde{X}$ can be recovered from $(\mathcal{E}, \kappa)$ as $\{e \in \mathcal{E} \mid \kappa(e^2) = 1\}$.

7.7.2 Let ${}^{rss}X^{(d)} \subset X^{(d)}$ be the open subscheme of divisors of the form $x_1 + \ldots + x_d$ with x_i pairwise distinct. Denote by ${}^{rss}\mathcal{S}_d \subset \mathcal{S}_d$ the preimage of ${}^{rss}X^{(d)}$ under the map $\mathcal{S}_d \to X^{(d)}$ sending a point of $\mathcal{S}_d$ to $\mathrm{div}(L^* \otimes \mathcal{C}/L)$. Set

$$\mathrm{RCov}^d = \mathrm{Bun}_1 \times_{\mathrm{Bun}_1} {}^{rss}X^{(d)},$$

where the map ${}^{rss}X^{(d)} \to \mathrm{Bun}_1$ sends D to $\mathcal{O}_X(-D)$, and the map $\mathrm{Bun}_1 \to \mathrm{Bun}_1$ takes a line bundle to its tensor square.

It is understood that ${}^{rss}X^{(0)} = \operatorname{Spec} k$ and the point ${}^{rss}X^{(0)} \to \mathrm{Bun}_1$ is $\mathcal{O}_X$.

Proposition 13. *The two-sheeted coverings $\pi : \tilde{X} \to X$ ramified exactly at $D \in {}^{rss}X^{(d)}$ (with $\tilde{X}$ assumed smooth) form an algebraic stack that can be identified with* RCov^d*.*

The stack ${}^{rss}\mathcal{S}_d$ classifies collections: $D \in {}^{rss}X^{(d)}$, a two-sheeted covering $\pi : \tilde{X} \to X$ ramified exactly at D, and a line bundle $\mathcal{B}$ on $\tilde{X}$.

Proof 1) Given a two-sheeted (ramified) covering $\pi : \tilde{X} \to X$ and a line bundle $\mathcal{B}$ on $\tilde{X}$ set $L = \pi_*\mathcal{B}$. Let σ be the nontrivial automorphism of $\tilde{X}$ over X. Let $x_1, \dots, x_d \in X$ be the points of the ramification and $\tilde{x}_1, \dots, \tilde{x}_d$ their preimages.

We have a canonical inclusion $\pi^*L \hookrightarrow \mathcal{B} \oplus \sigma^*\mathcal{B}$, actually

$$\pi^*L = \{v \in \mathcal{B} \oplus \sigma^*\mathcal{B} \mid \text{ the image of } v \text{ in } (\mathcal{B} \oplus \sigma^*\mathcal{B})_{\tilde{x}_i} \text{ lies in } \mathcal{B}_{\tilde{x}_i} \stackrel{\mathrm{diag}}{\to} (\mathcal{B} \oplus \sigma^*\mathcal{B})_{\tilde{x}_i} \text{ for all } i\}$$

In particular, $\pi^*(\det L) \widetilde{\to} \mathcal{B} \otimes \sigma^*\mathcal{B}(-\tilde{x}_1 - \dots - \tilde{x}_d)$.

Let $\mathcal{E}$ denote the σ-anti-invariants in $\pi_*\mathcal{O}$, so $\pi_*\mathcal{O} \widetilde{\to} \mathcal{O} \oplus \mathcal{E}$. Clearly, $\pi^*\mathcal{E} \widetilde{\to} \mathcal{O}(-\tilde{x}_1 - \dots - \tilde{x}_d)$, and σ acts on $\pi^*\mathcal{E}$ as -1. This yields an isomorphism

$$\kappa : \mathcal{E}^{\otimes 2} \widetilde{\to} \mathcal{O}(-x_1 - \dots - x_d)$$

The diagram

$$\begin{array}{ccc} \pi^* \operatorname{Sym}^2 L & \subset & \mathcal{B}^{\otimes 2} \oplus (\mathcal{B} \otimes \sigma^*\mathcal{B}) \oplus (\sigma^*\mathcal{B})^{\otimes 2} \\ & & \downarrow \\ \pi^*(\mathcal{E} \otimes \det L) & \subset & \mathcal{B} \otimes \sigma^*\mathcal{B} \end{array}$$

shows that $\pi^* \operatorname{Sym}^2 L \to \pi^*(\mathcal{E} \otimes \det L)(2\tilde{x}_1 + \dots + 2\tilde{x}_d)$ is regular and surjective. This map is compatible with the descent data, so gives rise to a regular surjective map

$$s : \operatorname{Sym}^2 L \to (\mathcal{E} \otimes \det L)(x_1 + \dots + x_d) \tag{16}$$

For each x_i on the fibre $L_{x_i} = \mathcal{B}/\mathcal{B}(-2\tilde{x}_i)$ we get a symmetric form whose kernel is exactly $\mathcal{B}(-\tilde{x}_i)/\mathcal{B}(-2\tilde{x}_i)$. Further, s induces an inclusion

$$L \hookrightarrow (\mathcal{E} \otimes L^* \otimes \det L)(x_1 + \dots + x_d) \widetilde{\to} L \otimes \mathcal{E}(x_1 + \dots + x_d)$$

and the quotient $(L \otimes \mathcal{E}(x_1 + \dots + x_d))/L$ is of length d.

For each x_i there is a base in $L \otimes \hat{\mathcal{O}}_{x_i}$ and in $\mathcal{E} \otimes \hat{\mathcal{O}}_{x_i}$ such that the matrix of $s : \operatorname{Sym}^2(\hat{\mathcal{O}}^2_{x_i}) \to \hat{\mathcal{O}}_{x_i}(x_i)$ over the formal disk at x_i becomes

$$\left(\begin{array}{cc} t^{-1} & 0 \\ 0 & 1 \end{array}\right),$$

where $t \in \hat{\mathcal{O}}_{x_i}$ is a local parameter. In other words,

$$(\mathcal{E} \otimes L(x_1 + \dots + x_d))/L \widetilde{\to} \mathcal{O}/\mathcal{O}(-x_1) \oplus \dots \oplus \mathcal{O}/\mathcal{O}(-x_d)$$

2) On the other side, let $s : \mathrm{Sym}^2 L \to \mathcal{C}$ be a k-point of $\mathcal{S}$ with $(L^* \otimes \mathcal{C})/L \widetilde{\to} \mathcal{O}_{x_1} \oplus \ldots \oplus \mathcal{O}_{x_d}$. Set $D = x_1 + \ldots + x_d$. Note that s is surjective. Let $\mathcal{I} \subset \mathrm{Sym}^* L$ denote the homogeneous ideal generated by the image of $s^* : \mathcal{C}^* \otimes (\det L)^2 \hookrightarrow \mathrm{Sym}^2 L$. Let $\tilde{X} \subset \mathbb{P}(L)$ denote the closed subscheme given by $\mathcal{I}$. Over $X - D$ this is exactly the curve of isotropic subspaces in L, as in Lemma 9. Write $\pi : \tilde{X} \to X$ for the projection.

We claim that $\tilde{X}$ is smooth. To check this in the neighbourhood of x_i, pick a base e_1, e_2 in $L \otimes \hat{\mathcal{O}}_{x_i}$ and $e \in \mathcal{C} \otimes \hat{\mathcal{O}}_{x_i}$ such that the matrix of $s : \mathrm{Sym}^2(L \otimes \hat{\mathcal{O}}_{x_i}) \to \mathcal{C} \otimes \hat{\mathcal{O}}_{x_i}$ in these bases becomes

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & t \end{array} \right),$$

where $t \in \hat{\mathcal{O}}_{x_i}$ is a local parameter. Then $\tilde{X} \times_X \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i}$ identifies with the closed subscheme $Y \stackrel{i}{\hookrightarrow} \mathbb{P}^1 \times \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i}$ given by $u_1^2 + t u_2^2 = 0$, where u_1, u_1 are the homogeneous coordinates on $\mathbb{P}^1$. The ring

$$\hat{\mathcal{O}}_{x_i}[u_1/u_2]/((u_1/u_2)^2 + t)$$

is a standard ramified extension of $\hat{\mathcal{O}}_{x_i}$ of degree 2. The scheme $\tilde{X} \times_X \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i}$ is regular, so $\tilde{X}$ is smooth. Note that $\pi^{-1}(x_i) =: \tilde{x}_i$ are exactly the ramification points of π.

Let $\mathcal{B}$ be the restriction of $\mathcal{O}_{\mathbb{P}(L)}(1)$ to $\tilde{X}$, it is equiped with $\pi^* L \to \mathcal{B}$. Let us check that the induced map $L \to \pi_* \mathcal{B}$ is an isomorphism. This is easy over $X - D$. Let $i : Y \hookrightarrow \mathbb{P}^1 \times \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i}$ be as above, $\xi : \mathbb{P}^1 \times \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i} \to \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i}$ be the projection. We must check that

$$\xi_* \mathcal{O}(1) \to \xi_* i_* i^* \mathcal{O}(1)$$

is an isomorphism. Define $\mathcal{V}$ by the exact sequence $0 \to \mathcal{V} \to \mathcal{O}(1) \to i_* i^* \mathcal{O}(1) \to 0$ on $\mathbb{P}^1 \times \mathrm{Spec}\, \hat{\mathcal{O}}_{x_i}$. It suffices to show that $\mathrm{R}^1 \xi_* \mathcal{V} = 0$. But this is easily checked fibrewise over $\hat{\mathcal{O}}_{x_i}$.

Let σ be the nontrivial automorphism of $\tilde{X}$ over X. Let $\mathcal{E}$ denote the σ-anti-invariants in $\pi_* \mathcal{O}$, so $\pi_* \mathcal{O} \widetilde{\to} \mathcal{O} \oplus \mathcal{E}$. By 1), we have $\mathcal{E}^{\otimes 2} \widetilde{\to} \mathcal{O}(-x_1 - \ldots - x_d)$ canonically, and L is equiped with the form (16). Define the vector bundle E on X by the exact sequence

$$0 \to E \to \mathrm{Sym}^2 L \stackrel{s}{\to} (\mathcal{E} \otimes \det L)(x_1 + \ldots + x_d)$$

As in Lemma 9, one checks that the composition $E \to \mathrm{Sym}^2 L \to \mathcal{C}$ vanishes, and the induced map $(\mathcal{E} \otimes \det L)(x_1 + \ldots + x_d) \to \mathcal{C}$ is an isomorphism. □

7.8 Local version $\mathcal{S}^{loc}$ of the stack $\mathcal{S}$

7.8.1 Set $\mathcal{O} = k[[t]]$ and $F = k((t))$. Let $\mathcal{S}^{loc}$ denote the stack classifying: a free $\mathcal{O}$-module L of rank 2, a free $\mathcal{O}$-module $\mathcal{C}$ of rank 1, and a map $s : \mathrm{Sym}^2 L \to \mathcal{C}$ inducing an inclusion $L \hookrightarrow L^* \otimes \mathcal{C}$.

Set $\mathrm{Sym}_+(\mathcal{O}) = \{B \in \mathrm{Mat}_2(\mathcal{O}) \mid {}^t B = B,\ \det B \neq 0\}$. This is a k-scheme not of finite type. Further, $\mathrm{GL}(2, \mathcal{O}) \times \mathcal{O}^*$ is a group scheme over k (not of finite type), and $\mathcal{S}^{loc}$ identifies with the stack quotient of $\mathrm{Sym}_+(\mathcal{O})$ by the action of $\mathrm{GL}(2, \mathcal{O}) \times \mathcal{O}^*$ given by $B \mapsto AB({}^t A)\epsilon$, $(A, \epsilon) \in \mathrm{GL}(2, \mathcal{O}) \times \mathcal{O}^*$.

Given a k-point $(L, \mathcal{C}, s)$ of $\mathcal{S}^{loc}$, there exist bases $e_1, e_2 \in L$ and $e \in \mathcal{C}$ such that the matrix of s in these bases is $\operatorname{diag}(t^a, t^b)$ for some $a \geq b \geq 0$ and

$$L^* \otimes \mathcal{C}/L \widetilde{\to} \mathcal{O}/t^a\mathcal{O} \oplus \mathcal{O}/t^b\mathcal{O}$$

It follows that two k-points $(L, \mathcal{C}, s)$ and $(L', \mathcal{C}', s')$ are isomorphic if and only if the $\mathcal{O}$-modules $L^* \otimes \mathcal{C}/L$ and $L'^* \otimes \mathcal{C}'/L'$ are isomorphic. We identify the set of isomorphism classes of k-points of $\mathcal{S}^{loc}$ with

$$\Phi = \{(a, b) \in \mathbb{Z}^2 \mid a \geq b \geq 0\}$$

For a closed point $x \in X$ a choice of an isomorphism $\mathcal{O} \widetilde{\to} \hat{\mathcal{O}}_{X,x}$ yields a map $\mathcal{S} \to \mathcal{S}^{loc}$ given by the restriction of $(L, \mathcal{C}, s)$ under $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$.

7.8.2 Denote by Cov_F the k-stack associating to a scheme S the groupoid of pairs (S', π), where S' is a scheme, and $\pi : S' \to S \times \operatorname{Spec} F$ is an étale covering of degree 2.

The stack Cov_F has (up to isomorphism) two k-points $(\operatorname{Spec} F', \pi)$, where the F-algebra F' is one of the following

- $F' \widetilde{\to} k((t^{\frac{1}{2}}))$ (anisotropic case)
- $F' \widetilde{\to} F \oplus F$ (hyperbolic case)

Given an S-point (S', π) of Cov_F, consider the rank 2 vector bundle $L = \pi_*\mathcal{O}_{S'}$ on $S \times \operatorname{Spec} F$. Let σ be the nontrivial automorphism of S' over $S \times \operatorname{Spec} F$. We have $L = \mathcal{O}_S \oplus \mathcal{E}$, where $\mathcal{E}$ denotes σ-anti-invariants in L. We have a canonical isomorphism $\kappa : \mathcal{E}^{\otimes 2} \widetilde{\to} \mathcal{O}_{S \times \operatorname{Spec} F}$. As in Remark 6, L is equiped with a symmetric form

$$\begin{array}{ccc} \operatorname{Sym}^2 L & \widetilde{\to} & \mathcal{O}_{S \times \operatorname{Spec} F} \oplus \mathcal{E} \oplus \mathcal{E}^{\otimes 2} \\ \downarrow s & \swarrow {\scriptstyle (1,0,-\kappa)} & \\ \mathcal{O}_{S \times \operatorname{Spec} F}, & & \end{array}$$

The form s in non degenerate, that is, induces an isomorphism $L \widetilde{\to} L^*$ of $\mathcal{O}_{S \times \operatorname{Spec} F}$-modules.

For a k-point of Cov_F, the symmetric form on L is either hyperbolic or anisotropic, this explains our terminology ([8], ch. 1). In the anisotrpic case $\kappa(t^{\frac{1}{2}} \otimes t^{\frac{1}{2}}) = t$.

It is easy to find a $\mathbb{A}^1$-point of Cov_F such that over $\mathbb{G}_m \subset \mathbb{A}^1$ we get the hyperbolic point of Cov_F and over $0 \in \mathbb{A}^1$ we get the anisotropic point.

We have a morphism of stacks $\mathcal{S}^{loc} \to \operatorname{Cov}_F$ defined as follows. If $(L, \mathcal{C}, \operatorname{Sym}^2 L \xrightarrow{s} \mathcal{C})$ is a S-point of $\mathcal{S}^{loc}$ then we have an isomorphism of vector bundles $L \widetilde{\to} (L^* \otimes \mathcal{C}) \mid_{S \times \operatorname{Spec} F}$ over $S \times \operatorname{Spec} F$. Define $S' \subset \mathbb{P}(L) \mid_{S \times \operatorname{Spec} F}$ as the closed subscheme corresponding to the homogeneous ideal in $\operatorname{Sym}^2(L \otimes F)$ generated by the image of

$$\mathcal{C}^* \otimes (\det L)^2 \otimes F \to \operatorname{Sym}^2(L \otimes F)$$

Then $\pi : S' \to S \times \operatorname{Spec} F$ is a point of Cov_F.

The image of the k-point $(a, b) \in \Phi$ under $\mathcal{S}^{loc} \to \operatorname{Cov}_F$ is anysotropic if $a - b$ is odd and hyperbolic otherwise.

7.9 For $d, k \geq 0$ let

$$\xi_{d,k} : {}^{rss}\mathcal{S}_{d,k} \to {}^{rss}\mathcal{S}_d$$

denote the stack over ${}^{rss}\mathcal{S}_d$ classifying: a point of ${}^{rss}\mathcal{S}_d$ given by $(L, \mathcal{C}, \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C})$, a subsheaf $L' \subset L$, where L/L' is a torsion sheaf of length k on X, such that the composition

$$\mathrm{Sym}^2 L' \to \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C}$$

is surjective.

We have a morphism of stack ${}^{rss}\mathcal{S}_{d,k} \times X^{(m)} \to \mathcal{S}$ sending $(L' \subset L, \mathcal{C}, \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C},\ D' \in X^{(m)})$ to $(L', \mathcal{C}(D'), s')$, where s' is the composition

$$\mathrm{Sym}^2 L' \to \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C} \hookrightarrow \mathcal{C}(D')$$

Proposition 14. *The stacks ${}^{rss}\mathcal{S}_{d,k} \times X^{(m)}$ form a stratification of $\mathcal{S}$.*

Proof Recall that a point of $\mathcal{S}$ is given by $(L, \mathcal{C}, \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C})$. Let $\mathcal{S}^0 \subset \mathcal{S}$ be the open substack given by the condition that s is surjective. Sratifying $\mathcal{S}$ by length of the cokernel of s, we are reduced to show that ${}^{rss}\mathcal{S}_{d,k}$ form a stratification of $\mathcal{S}^0$.

Let $(L', \mathcal{C}, s)$ be a k-point of $\mathcal{S}^0$. Set $D = \mathrm{div}((L'^* \otimes \mathcal{C})/L')$ and write $D = \sum d_x x$. The restriction of $(L'^* \otimes \mathcal{C})/L'$ to $\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x}$ is isomorphic to $\mathcal{O}/t_x^{d_x}\mathcal{O}$, where t_x is a local parameter at x. There is a unique subsheaf $L' \subset L \subset L'^* \otimes \mathcal{C}$ such that s extends to a map $\mathrm{Sym}^2 L \to \mathcal{C}$ yielding

$$L' \subset L \subset L^* \otimes \mathcal{C} \subset L'^* \otimes \mathcal{C},$$

and

$$(L^* \otimes \mathcal{C})/L \mid_{\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x}} \;\widetilde{\to}\; \begin{cases} 0, & \text{if } d_x \text{ is even} \\ k, & \text{if } d_x \text{ is odd} \end{cases}$$

Our assertion follows. $\square$

7.10 The stack $\mathcal{S}_\pi$

Fix a k-point of RCov^d given by $D_\pi \in {}^{rss}X^{(d)}$ and $\pi : \tilde{X} \to X$ ramified exactly at D_π.

Given a point $(L, \mathcal{C}, \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C})$ of $\mathcal{S}$, set

$$D = \mathrm{div}(L^* \otimes \mathcal{C}/L)$$

and let $\pi_L : \tilde{X}_L \to X - D$ denote the corresponding two-sheeted covering defined as in Lemma 9. Denote by $\mathcal{S}_\pi$ the stack classifying: a point $(L, \mathcal{C}, \mathrm{Sym}^2 L \xrightarrow{s} \mathcal{C})$ of $\mathcal{S}$ together with an isomorphism over $X - D$

$$\begin{array}{ccc} \tilde{X}_L & \widetilde{\to} & \pi^{-1}(X - D) \\ \downarrow \pi_L & \swarrow & \\ X - D & & \end{array}$$

(note that D_π does not intersect $X - D$, because π_L is unramified).

7.10.1 Let $\tilde{E}$ be a rank one local system on $\tilde{X}$. We are going to define the category $\mathrm{P}^{\tilde{E}}(\mathcal{S}_\pi)$ of $\tilde{E}$-equivariant perverse sheaves on $\mathcal{S}_\pi$.

Let $(X\times\mathcal{S}_\pi)^0\subset X\times\mathcal{S}_\pi$ be the open substack of those $x\in X$, $(L,\mathcal{C},\mathrm{Sym}^2 L\to\mathcal{C})\in\mathcal{S}_\pi$, for which the map $L\to L^*\otimes\mathcal{C}$ is an isomorphism over the formal disk around $x\in X$.

Let $(\tilde{X}\times\mathcal{S}_\pi)^0$ denote the preimage of $(X\times\mathcal{S}_\pi)^0$ under

$$\pi\times\mathrm{id}:\tilde{X}\times\mathcal{S}_\pi\to X\times\mathcal{S}_\pi$$

Write $\mathcal{H}_{\mathcal{S}_\pi}$ for the stack classifying: a point of $(X\times\mathcal{S}_\pi)^0$ given by $(L,\mathcal{C},\mathrm{Sym}^2 L\stackrel{s}{\to}\mathcal{C})\in\mathcal{S}_\pi$, $x\in X$ together with a commutative diagram

$$\begin{array}{ccc}\mathrm{Sym}^2 L' & \to & \mathcal{C}(x)\\ \cup & & \cup\\ \mathrm{Sym}^2 L & \stackrel{s}{\to} & \mathcal{C},\end{array}$$

where $L\subset L'\subset L(x)$ is an upper modification of L with $x=\mathrm{div}(L'/L)$.

We have a diagram

$$(\tilde{X}\times\mathcal{S}_\pi)^0\stackrel{\mathrm{supp}\times\mathfrak{p}_{\mathcal{S}_\pi}}{\leftarrow}\mathcal{H}_{\mathcal{S}_\pi}\stackrel{\mathfrak{q}_{\mathcal{S}_\pi}}{\to}\mathcal{S}_\pi,$$

where $\mathrm{supp}\times\mathfrak{p}_{\mathcal{S}_\pi}$ sends a point of $\mathcal{H}_{\mathcal{S}_\pi}$ to $(L,\mathcal{C},\mathrm{Sym}^2 L\stackrel{s}{\to}\mathcal{C})\in\mathcal{S}_\pi$ together with the point $\tilde{x}\in\tilde{X}$ corresponding to the isotropic subspace $L'/L\subset L(x)/L$, so $\pi(\tilde{x})=x$. Actually, $\mathrm{supp}\times\mathfrak{p}_{\mathcal{S}_\pi}$ is an isomorphism. The map $\mathfrak{q}_{\mathcal{S}_\pi}$ sends the above point to

$$(L',\mathcal{C}(x),\mathrm{Sym}^2 L'\to\mathcal{C}(x))\in\mathcal{S}_\pi$$

The following is a version of the *Waldspurger category* that will be introduced in Sect. 8.

Definition 4. Let $\mathrm{P}^{\tilde{E}}(\mathcal{S}_\pi)$ be the category, whose objects are pairs: a perverse sheaf F on $\mathcal{S}_\pi$ and an isomorphism

$$(\mathrm{supp}\times\mathfrak{p}_{\mathcal{S}_\pi})_!\mathfrak{q}^*_{\mathcal{S}_\pi}F\,\widetilde{\to}\,\tilde{E}\boxtimes F$$

over $(\tilde{X}\times\mathcal{S}_\pi)^0$. The morphisms in $\mathrm{P}(\mathcal{S}_\pi)$ are the maps of the corresponding perverse sheaves compatible with the equivariance isomorphisms.

8. Waldspurger model for GL_2

8.1 Fix a k-point of RCov^d given by $D_\pi\in{}^{rss}X^{(d)}$ and $\pi:\tilde{X}\to X$ ramified exactly at D_π. Denote by σ the nontrivial automorphism of $\tilde{X}$ over X, let $\mathcal{E}$ be the σ-anti-invariants in $\pi_*\mathcal{O}_{\tilde{X}}$.

Fix a k-point $x\in X$, write $\mathcal{O}_x$ for the completed local ring of X at x and F_x for its fraction field. Write $\tilde{F}_x$ for the étale F_x-algebra of regular functions on $\tilde{X}\times_X\mathrm{Spec}\,F_x$. If $x\in D_\pi$ then $\tilde{F}_x$ is anysotropic otherwise it is hyperbolic (cf. Sect. 7.8.2). Denote by $\tilde{\mathcal{O}}_x$ the ring of regular functions on $\tilde{X}\times_X\mathrm{Spec}\,\mathcal{O}_x$.

Definition 5. Let $\mathcal{W}\mathrm{ald}_\pi^{x,loc}$ denote the stack classifying: a free $\mathcal{O}_x$-module L of rank 2, a free $\tilde{F}_x$-module $\mathcal{B}$ of rank 1 together with an isomorphism $\xi:L\otimes_{\mathcal{O}_x}F_x\,\widetilde{\to}\,\mathcal{B}$ of F_x-modules.

Let $\mathrm{GL}(\tilde{F}_x)$ denote the group of automorphisms of the F_x-linear vector space $\tilde{F}_x$, let $\mathrm{GL}(\tilde{\mathcal{O}}_x) \subset \mathrm{GL}(\tilde{F}_x)$ be the stabilizor of $\tilde{\mathcal{O}}_x$. Then $\mathcal{W}\mathrm{ald}_\pi^{x,loc}$ identifies with the stack quotient of the affine grassmanian $\mathrm{Gr}_{\tilde{F}_x} := \mathrm{GL}(\tilde{F}_x)/\mathrm{GL}(\tilde{\mathcal{O}}_x)$ by the group ind-scheme $\tilde{F}_x^*$.

A choice of a base in the free $\mathcal{O}_x$-module $\tilde{\mathcal{O}}_x$ yields isomorphisms $\mathrm{GL}(\tilde{F}_x) \widetilde{\to} \mathrm{GL}_2(F_x)$, $\mathrm{GL}(\tilde{\mathcal{O}}_x) \widetilde{\to} \mathrm{GL}_2(\mathcal{O}_x)$, and an inclusion $\tilde{F}_x^* \hookrightarrow \mathrm{GL}_2(F_x)$.

For a k-point of $\mathcal{W}\mathrm{ald}_\pi^{x,loc}$ consider the set of free $\tilde{\mathcal{O}}_x$-submodules of rank one $\mathcal{B}_{ex} \subset \mathcal{B}$ such that $\xi(L) \subset \mathcal{B}_{ex}$. This set contains a unique minimal element that we denote by $\mathcal{B}_{ex}$.

In both split ($x \notin D_\pi$) and nonsplit ($x \in D_\pi$) case the isomorphism classes of k-points of $\mathcal{W}\mathrm{ald}_\pi^{x,loc}$ are indexed by non negative integers $m \geq 0$, the corresponding point is given by $\deg(\mathcal{B}_{ex}/L) = m$. Denote by $\mathrm{Gr}^m_{\tilde{F}_x}$ the $\tilde{F}_x^*$-orbit on $\mathrm{Gr}_{\tilde{F}_x}$ corresponding to $m \geq 0$.

In matrix terms, in the split case $\tilde{\mathcal{O}}_x \widetilde{\to} \mathcal{O}_x \oplus \mathcal{O}_x$ has a distinguished (defined up to permutation) base $\{(1,0),(0,1)\}$ over $\mathcal{O}_x$. This base yields an inclusion $\tilde{F}_x^* \hookrightarrow \mathrm{GL}_2(F_x)$ whose image is the set of diagonal matrices. Then $\tilde{F}_x^*$-orbit on $\mathrm{GL}_2(F_x)/\mathrm{GL}_2(\mathcal{O}_x)$ corresponding to $m \geq 0$ is given by the matrix

$$\left(\begin{array}{cc} t^m & 1 \\ 0 & 1 \end{array}\right),$$

where $t \in \mathcal{O}_x$ is a local parameter (cf. [3], Sect. 1).

In the nonsplit case the lattice $\mathcal{O}_x \oplus \mathcal{O}_x t^{m+\frac{1}{2}} \subset \tilde{F}_x$ is a representative for the $\tilde{F}_x^*$-orbit on $\mathrm{Gr}_{\tilde{F}_x}$ corresponding to $m \geq 0$. Here $t \in \mathcal{O}_x$ is a local parameter.

8.2 In the same manner as in [4] we can consider the following global model of $\mathcal{W}\mathrm{ald}_\pi^{x,loc}$.

Definition 6. Let $\mathcal{W}\mathrm{ald}_\pi^x$ denote the stack classifying: a rank 2 vector bundle L on X, a line bundle $\mathcal{B}$ on $\pi^{-1}(X-x)$ and an isomorphism $L \widetilde{\to} \pi_*\mathcal{B}$ over $X - x$.

As in Proposition 13, a point of $\mathcal{W}\mathrm{ald}_\pi^x$ gives rise to a map

$$s : \mathrm{Sym}^2 L \to (\mathcal{E} \otimes \det L)(D_\pi + \infty x)$$

Write $\mathcal{W}\mathrm{ald}_\pi^{x,\leq m} \hookrightarrow \mathcal{W}\mathrm{ald}_\pi^x$ for the closed substack given by the condition that

$$s : \mathrm{Sym}^2 L \to (\mathcal{E} \otimes \det L)(D_\pi + mx) \tag{17}$$

is regular.

Lemma 10. *The stack $\mathcal{W}\mathrm{ald}_\pi^{x,\leq m}$ is algebraic, so $\mathcal{W}\mathrm{ald}_\pi^x$ is an inductive limit of algebraic stacks.*

Proof Set $\overline{\mathrm{RCov}}^d = \mathrm{Bun}_1 \times_{\mathrm{Bun}_1} X^{(d)}$, where the map $X^{(d)} \to \mathrm{Bun}_1$ sends D to $\mathcal{O}_X(-D)$ and $\mathrm{Bun}_1 \to \mathrm{Bun}_1$ takes a line bundle to its tensor square. We have a map $\mathcal{S}_d \to \overline{\mathrm{RCov}}^d$ sending $(L, \mathrm{Sym}^2 L \to \mathcal{C})$ to $(\det L) \otimes \mathcal{C}^{-1}$ equiped with $(\det L)^{\otimes 2} \otimes \mathcal{C}^{\otimes -2} \widetilde{\to} \mathcal{O}(-D)$, $D \in X^{(d)}$.

For $d = \deg D_\pi + 2m$ consider the k-point $(\mathcal{E}(-mx), (\mathcal{E}(-mx))^{\otimes 2} \widetilde{\to} \mathcal{O}(-D_\pi - 2mx))$ of $\overline{\mathrm{RCov}}^d$. Then $\mathcal{W}\mathrm{ald}_\pi^{x,\leq m}$ is the fibre of $\mathcal{S}_d \to \overline{\mathrm{RCov}}^d$ over this k-point. $\square$

Denote by

$$\mathcal{W}\mathrm{ald}_\pi^{x,m} \hookrightarrow \mathcal{W}\mathrm{ald}_\pi^{x,\le m} \tag{18}$$

the open substack given by the condition that (17) is surjective. The stack $\mathcal{W}\mathrm{ald}_\pi^{x,m}$ classifies collections: a line bundle $\mathcal{B}_{ex}$ on $\tilde{X}$, for which we set $L_{ex} = \pi_*\mathcal{B}_{ex}$, and a lower modification $L \subset L_{ex}$ of vector bundles on X such that the composition is surjective

$$\operatorname{Sym}^2 L \to \operatorname{Sym}^2 L_{ex} \xrightarrow{s} \mathcal{C}$$

and $\operatorname{div}(L_{ex}/L) = mx$. Here we have denoted $\mathcal{C} = (\mathcal{E} \otimes \det L_{ex})(D_\pi)$, so $(L_{ex}, \mathcal{C}, \operatorname{Sym}^2 L_{ex} \xrightarrow{s} \mathcal{C})$ is the point of ${}^{rss}\mathcal{S}$ corresponding to $\mathcal{B}_{ex}$.

Another way to say is that the stratum $\mathcal{W}\mathrm{ald}_\pi^{x,m}$ is given by fixing an extension of $\mathcal{B}$ to a line bundle $\mathcal{B}_{ex}$ on $\tilde{X}$ such that for $L_{ex} := \pi_*\mathcal{B}_{ex}$ we have $L \subset L_{ex}$ and $\mathcal{B}_{ex}$ is the smallest with this property. Then $L_{ex}/L \widetilde{\to} \mathcal{O}_x/t^m$, where $t \in \mathcal{O}_x$ is a local parameter.

Denote by $\mathrm{pr}_{\mathcal{W}} : \mathcal{W}\mathrm{ald}_\pi^{x,m} \to \operatorname{Pic} \tilde{X}$ the map sending the above point to $\mathcal{B}_{ex}$.

8.2.1 Here is one more description. Denote by $(\operatorname{Pic} \tilde{X})^x$ the scheme classifying a line bundle $\mathcal{B}_{ex}$ on $\tilde{X}$ together with a trivialization $\mathcal{B} \otimes \tilde{\mathcal{O}}_x \widetilde{\to} \tilde{\mathcal{O}}_x$. The group $\tilde{\mathcal{O}}_x^*$ acts on $(\operatorname{Pic} \tilde{X})^x$ by changing the trivialization. It is well-known that this action extends to an action of the group ind-scheme $\tilde{F}_x^*$ on $(\operatorname{Pic} \tilde{X})^x$.

Consider the action of $\tilde{F}_x^*$ on $(\operatorname{Pic} \tilde{X})^x \times \mathrm{Gr}_{\tilde{F}_x}$ which is the product of natural actions on the factors. Then $\mathcal{W}\mathrm{ald}_\pi^x$ identifies with the stack quotient of $(\operatorname{Pic} \tilde{X})^x \times \mathrm{Gr}_{\tilde{F}_x}$ by $\tilde{F}_x^*$. Let $f_{\mathcal{W}} : (\operatorname{Pic} \tilde{X})^x \times \mathrm{Gr}_{\tilde{F}_x} \to \mathcal{W}\mathrm{ald}_\pi^x$ be the corresponding map.

8.3 Fix a rank one local system $\tilde{E}$ on $\tilde{X}$. The $\tilde{\mathcal{O}}_x^*$-orbits on $\mathrm{Gr}_{\tilde{F}_x}$ are finite-dimensional. So, we have the category of $\tilde{\mathcal{O}}_x^*$-equivariant perverse sheaves on $\mathrm{Gr}_{\tilde{F}_x}$.

Definition 7. *Waldspurger category* $\mathrm{P}^{\tilde{E}}(\mathrm{Gr}_{\tilde{F}_x})$ is the category of those $\tilde{\mathcal{O}}_x^*$-equivariant perverse sheaves on $\mathrm{Gr}_{\tilde{F}_x}$ that

- (the nonsplit case) under the action of a uniformazer $\in \tilde{F}_x^*/\tilde{\mathcal{O}}_x^*$ change by $\tilde{E}_{\tilde{x}}$, where $\pi(\tilde{x}) = x$.
- (the split case) under the action of a uniformizer $t_{\tilde{x}} \in \tilde{F}_x^*/\tilde{\mathcal{O}}_x^*$ change by $\tilde{E}_{\tilde{x}}$ for both $\tilde{x} \in \pi^{-1}(x)$.

One should be carefull about the following. Though $\mathrm{P}^{\tilde{E}}(\mathrm{Gr}_{\tilde{F}_x})$ is a full subcategory of the category $\mathrm{P}(\mathrm{Gr}_{\tilde{F}_x})$ of perverse sheaves on $\mathrm{Gr}_{\tilde{F}_x}$, the Ext groups in these two categories may be different. This is due to the fact that the $\tilde{\mathcal{O}}_x^*$-orbits on $\mathrm{Gr}_{\tilde{F}_x}$ are not contractible.

Denote by $A\tilde{E}$ the automorphic local system on $\operatorname{Pic} \tilde{X}$ corresponding to $\tilde{E}$. For $d \ge 0$ its inverse image under $\tilde{X}^{(d)} \to \operatorname{Pic}^d \tilde{X}$ identifies with the symmetric power $\tilde{E}^{(d)}$ of $\tilde{E}$. Define the perverse sheaf $\mathcal{W}_m$ on $\mathcal{W}\mathrm{ald}_\pi^x$ as the Goresky-MacPherson extension of

$$\mathrm{pr}_{\mathcal{W}}^* A\tilde{E} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{W}\mathrm{ald}_\pi^{x,m}}$$

under (18).

For any k-point of $\mathrm{Gr}_{\tilde{F}_x}$ its stabilizor in $\tilde{F}_x^*$ is connected. So, the irreducible objects of $\mathrm{P}^{\tilde{E}}(\mathrm{Gr}_{\tilde{F}_x})$ are indexed by $m\ge 0$, the irreducibe object $\tilde{\mathcal{W}}_m\in \mathrm{P}^{\tilde{E}}(\mathrm{Gr}_{\tilde{F}_x})$, defined up to a scalar automorphism, can be described by the following property: for the diagram

$$\mathcal{W}\mathrm{ald}_\pi^x \stackrel{f_{\mathcal{W}}}{\leftarrow} (\operatorname{Pic}\tilde{X})^x\times \mathrm{Gr}_{\tilde{F}_x} \stackrel{\mathrm{pr}\times\mathrm{id}}{\to} \operatorname{Pic}\tilde{X}\times \mathrm{Gr}_{\tilde{F}_x}$$

we have $(\mathrm{pr}^* A\tilde{E})\boxtimes\tilde{\mathcal{W}}_m \widetilde{\to} f_{\mathcal{W}}^*\mathcal{W}_m$.

The group scheme $(\operatorname{Pic}\tilde{E})^x$ acts on $\mathcal{W}\mathrm{ald}_\pi^x$ as follows. The action map

$$\mathrm{act}: (\operatorname{Pic}\tilde{E})^x\times \mathcal{W}\mathrm{ald}_\pi^x\to \mathcal{W}\mathrm{ald}_\pi^x$$

sends $(\mathcal{B}',\, \nu:\mathcal{B}'\otimes\tilde{\mathcal{O}}_x\widetilde{\to}\tilde{\mathcal{O}}_x)\in(\operatorname{Pic}\tilde{E})^x$ and $(\mathcal{B}, L, \pi_*\mathcal{B}\widetilde{\to} L\mid_{X-x})\in \mathcal{W}\mathrm{ald}_\pi^x$ to

$$(\mathcal{B}\otimes\mathcal{B}', \pi_*(\mathcal{B}\otimes\mathcal{B}')\widetilde{\to} L'\mid_{X-x})\in \mathcal{W}\mathrm{ald}_\pi^x,$$

where the vector bundle L' on X is the gluing of $\pi_*(\mathcal{B}\otimes\mathcal{B}')\mid_{X-x}$ and $L\mid_{\operatorname{Spec}\mathcal{O}_x}$ via the isomorphism $(\pi_*(\mathcal{B}\otimes\mathcal{B}'))\otimes F_x\widetilde{\to} L\otimes F_x$ induced by ν.

Let $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}_\pi^x)$ be the category of perverse sheaves on $\mathcal{W}\mathrm{ald}_\pi^x$ that change by $\mathrm{pr}^*\tilde{E}$ under the action of $(\operatorname{Pic}\tilde{E})^x$, where $\mathrm{pr}:(\operatorname{Pic}\tilde{E})^x\to\operatorname{Pic}\tilde{E}$ is the projection.

Here is one more description of this category. Let

$$\mathfrak{q}_{\mathcal{W}\mathrm{ald}}:\pi^{-1}(X-x)\times \mathcal{W}\mathrm{ald}_\pi^x\to \mathcal{W}\mathrm{ald}_\pi^x$$

be the map sending $(\tilde{x},\mathcal{B},\pi_*\mathcal{B}\widetilde{\to} L\mid_{X-x})$ to $(\mathcal{B}(\tilde{x}),\pi_*\mathcal{B}(\tilde{x})\widetilde{\to} L'\mid_{X-x})$, where the vector bundle L' on X is the gluing of $\pi_*\mathcal{B}(\tilde{x})\mid_{X-x}$ and $L\otimes\mathcal{O}_x$ via the isomorphism $(\pi_*\mathcal{B}(\tilde{x}))\otimes F_x\widetilde{\to} L\otimes F_x$, which is due to the fact that $\pi(\tilde{x})\ne x$.

Then $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}_\pi^x)$ is equivalent to the category of pairs: a perverse sheaf F on $\mathcal{W}\mathrm{ald}_\pi^x$ and an isomorphism $\mathfrak{q}_{\mathcal{W}\mathrm{ald}}^*F\widetilde{\to}\tilde{E}\boxtimes F$.

The irreducible objects of $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}_\pi^x)$ are exactly $\mathcal{W}_m$, $m\ge 0$.

8.4 Let $\mathrm{Sph}(\mathrm{Gr}_{\mathrm{GL}_2})$ be the category of $\mathrm{GL}_2(\mathcal{O}_x)$-equivariant (spherical) perverse sheaves on the affine grassmanian $\mathrm{Gr}_{\mathrm{GL}_2}$. This is a tensor category equivalent to the category of representations of GL_2 over $\bar{\mathbb{Q}}_\ell$ ([7]). It acts on $\mathrm{D}(\mathcal{W}\mathrm{ald}_\pi^x)$ by Hecke functors as follows.

Let ${}_x\mathcal{H}_{\mathrm{GL}_2}$ denote the Hecke stack classifying vector bundles L, L' on X together with an isomorphism $\beta: L\widetilde{\to} L'\mid_{X-x}$ over $X-x$. Consider the diagram

$$\mathcal{W}\mathrm{ald}_\pi^x \stackrel{\mathfrak{p}_{\mathcal{W}}}{\leftarrow} \mathcal{W}\mathrm{ald}_\pi^x\times_{\mathrm{Bun}_2}{}_x\mathcal{H}_{\mathrm{GL}_2} \stackrel{\mathfrak{q}_{\mathcal{W}}}{\to} \mathcal{W}\mathrm{ald}_\pi^x,$$

where $\mathfrak{p}_{\mathcal{W}}$ sends a collection $(L, L', \beta, \mathcal{B}, \pi_*\mathcal{B}\widetilde{\to} L\mid_{X-x})$ to $(L,\mathcal{B},\pi_*\mathcal{B}\widetilde{\to} L\mid_{X-x})$ and $\mathfrak{q}_{\mathcal{W}}$ sends this collection to $(L',\mathcal{B},\pi_*\mathcal{B}\widetilde{\to} L'\mid_{X-x})$.

Let Bun_2^x be the stack classifying $L\in\mathrm{Bun}_2$ together with its trivialization over $\operatorname{Spec}\mathcal{O}_x$. The projection $\mathfrak{q}_{\mathrm{GL}_2}: {}_x\mathcal{H}_{\mathrm{GL}_2}\to\mathrm{Bun}_2$ forgetting L can be realized as a fibration

$$\mathrm{Bun}_2^x\times_{\mathrm{GL}_2(\mathcal{O}_x)}\mathrm{Gr}_{\mathrm{GL}_2}\to\mathrm{Bun}_2,$$

so for $K \in \mathrm{D}(\mathcal{W}\mathrm{ald}^x_\pi)$ and $\mathcal{A} \in \mathrm{Sph}(\mathrm{Gr}_{\mathrm{GL}_2})$ we may form the corresponding twisted exterior product $K\tilde{\boxtimes}\mathcal{A}$. It is normilized so that it is perverse for K perverse and

$$\mathbb{D}(K\tilde{\boxtimes}\mathcal{A}) \widetilde{\to} \mathbb{D}(K)\tilde{\boxtimes}\,\mathbb{D}(\mathcal{A})$$

Let $\mathrm{H}(\mathcal{A},\cdot) : \mathrm{D}(\mathcal{W}\mathrm{ald}^x_\pi) \to \mathrm{D}(\mathcal{W}\mathrm{ald}^x_\pi)$ be the functor given by

$$\mathrm{H}(\mathcal{A}, K) = (\mathfrak{p}_{\mathcal{W}})_!(K\tilde{\boxtimes}\mathcal{A})$$

These functors are compatible with the tensor structure on $\mathrm{Sph}(\mathrm{Gr}_{\mathrm{GL}_2})$ in the sense that we have isomorphisms

$$\mathrm{H}(\mathcal{A}_1, \mathrm{H}(\mathcal{A}_2, K)) \widetilde{\to} \mathrm{H}(\mathcal{A}_1 * \mathcal{A}_2, K), \tag{19}$$

where $\mathcal{A}_1 * \mathcal{A}_2 \in \mathrm{Sph}(\mathrm{Gr}_{\mathrm{GL}_2})$ is the convolution (cf. [4], Sect. 5). One checks that $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}^x_\pi)$ is preserved by Hecke functors.

Theorem 4. *1) For $d \geq 0$ let $\lambda = (d, 0) \in \Lambda^+_{\mathrm{GL}_2}$. We have a canonical isomorphism*

$$\mathrm{H}(\mathcal{A}_\lambda, \mathcal{W}_0) \widetilde{\to} \mathcal{W}_d$$

2) For $\lambda = (1, 1)$ and $d \geq 0$ we have canonically

$$\mathrm{H}(\mathcal{A}_\lambda, \mathcal{W}_d) \widetilde{\to} \begin{cases} \mathcal{W}_d \otimes \tilde{E}^{\otimes 2}_{\tilde{x}}, & \textit{the nonsplit case, } \pi(\tilde{x}) = x \\ \\ \mathcal{W}_d \otimes \tilde{E}_{\tilde{x}_1} \otimes \tilde{E}_{\tilde{x}_2}, & \textit{the split case, } \pi^{-1}(x) = \{x_1, x_2\} \end{cases}$$

8.5 Set $\Lambda^+_{\mathrm{GL}_2} = \{(a_1 \geq a_2) \mid a_i \in \mathbb{Z}\}$. We view $\Lambda^+_{\mathrm{GL}_2}$ as the set of dominant coweights for GL_2. For $\lambda = (a_1, a_2) \in \Lambda^+_{\mathrm{GL}_2}$ denote by $\mathrm{Gr}^\lambda_{\tilde{F}_x} \subset \mathrm{Gr}_{\tilde{F}_x}$ the locally closed subscheme classifying $\mathcal{O}_x$-sublattices $L \subset t^{a_2}\tilde{\mathcal{O}}_x$ such that

$$t^{a_2}\tilde{\mathcal{O}}_x/L \widetilde{\to} \mathcal{O}_x/t^{a_1 - a_2}$$

as $\mathcal{O}_x$-modules. Let $\overline{\mathrm{Gr}}^\lambda_{\tilde{F}_x}$ denote the closure of $\mathrm{Gr}^\lambda_{\tilde{F}_x}$ in $\mathrm{Gr}_{\tilde{F}_x}$.

Our proof of Theorem 4 is inspired by ([4], Theorem 4), the following is a key point.

Proposition 15. *For $m \geq 0$ and a dominant coweight $\lambda = (a_1 \geq a_2)$ of GL_2 the intersection $\overline{\mathrm{Gr}}^\lambda_{\tilde{F}_x} \cap \mathrm{Gr}^m_{\tilde{F}_x}$ is non empty iff $0 \leq m \leq a_1 - a_2$ and has pure dimension m.*

Proof 1) (the split case). Use the matrix realization of $\mathrm{Gr}_{\tilde{F}_x}$ as in Sect. 8.1. Using the action of the center of GL_2, we may reduce to the case $\lambda = (a, 0)$. Stratify $\overline{\mathrm{Gr}}^\lambda_{\mathrm{GL}_2}$ by intersecting with $N(F_x)$-orbits on the affine grassmanian, where $N \subset \mathrm{GL}_2$ is the standard maximal unipotent subgroup. For all strata the argument is the same, let us explain it for the open stratum

$$\left\{ \begin{pmatrix} t^a & b \\ 0 & 1 \end{pmatrix}, b \in \mathcal{O}_x \right\} / \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, c \in \mathcal{O}_x \right\},$$

which we identify with $\mathcal{O}_x/t^a$ via the map $\begin{pmatrix} t^a & b \\ 0 & 1 \end{pmatrix} \mapsto b$. The point $b \in \mathcal{O}/t^a$ lies in $\mathrm{Gr}^m_{\tilde F_x}$ iff $b \in t^{a-m}\mathcal{O}^*_x$.

2) (the nonsplit case). Let $t \in \mathcal{O}_x$ be a local parameter. Multiplying by an appropiate power of t we are reduced to the case $\lambda = (a, 0)$. Then $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x}$ is the scheme of $\mathcal{O}_x$-sublattices $L \subset \tilde{\mathcal{O}}_x$ such that $\dim(\tilde{\mathcal{O}}_x/L) = d$. The intersection $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x} \cap \mathrm{Gr}^m_{\tilde F_x}$ is then the scheme of sublattices

$$L \subset t^{\frac{1}{2}(a-m)}\tilde{\mathcal{O}}_x \subset \tilde{\mathcal{O}}_x$$

such that $\dim(t^{\frac{1}{2}(a-m)}\tilde{\mathcal{O}}_x)/L = m$ and $L \not\subset t^{\frac{a-m+1}{2}}\tilde{\mathcal{O}}_x$. Our assertion follows. $\square$

Remark 7. In the nonsplit case the schemes $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x} \cap \mathrm{Gr}^m_{\tilde F_x}$ are connected, whence in the split case they admit several connected components.

Actually, we need the following a bit different result. Given $\lambda \in \Lambda^+_{\mathrm{GL}_2}$ and $\mathcal{O}_x$-lattice $L \subset \tilde F_x$, denote by $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x}(L) \subset \mathrm{Gr}_{\tilde F_x}$ the closed subscheme of $\mathcal{O}_x$-lattices $L' \subset \tilde F_x$ such that

$$(L', L,\ L \otimes F_x \widetilde{\to} L' \otimes F_x) \in \overline{\mathrm{Gr}}^\lambda_{\mathrm{GL}_2}(L)$$

More precisely, for any isomorphism $L \widetilde{\to} \mathcal{O}_x \oplus \mathcal{O}_x$ of $\mathcal{O}_x$-modules the corresponding point

$$(L', L' \otimes F_x \widetilde{\to} F_x \oplus F_x) \in \overline{\mathrm{Gr}}^\lambda_{\mathrm{GL}_2}$$

Proposition 16. *Let $m \geq 0$ and $L \subset \tilde F_x$ be a $\mathcal{O}_x$-lattice lying in $\mathrm{Gr}^m_{\tilde F_x}$. For a dominant coweight $\lambda = (d, 0)$ of GL_2 the intersection $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x}(L) \cap \mathrm{Gr}^0_{\tilde F_x}$ is empty unless $d \geq m$. For $d \geq m$ it is a point (resp., a union of $d - m + 1$ points) in the nonsplit case (resp., in the split case).*

Proof 1) (the nonsplit case). Multiplying by a suitable element of $\tilde F^*_x$, we may assume $L = \mathcal{O}_x \oplus \mathcal{O}_x t^{m+\frac{1}{2}}$. The scheme $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x}(L)$ classifies $\mathcal{O}_x$-sublattices $L' \subset L$ such that $\dim(L/L') = d$. A point L' of this scheme lies in $\mathrm{Gr}^0_{\tilde F_x}$ if and only if $L' = t^{\frac{d+m}{2}}\tilde{\mathcal{O}}_x$. Our assertion follows.

2) (the split case). Choose a base $\{e_1, e_2\}$ in $\tilde{\mathcal{O}}_x$ over $\mathcal{O}_x$ as in 8.1. Multiplying by an suitable element of $\tilde F^*_x$ we may assume

$$L = t^m\mathcal{O}_x e_1 \oplus \mathcal{O}_x(e_1 + e_2)$$

The scheme $\overline{\mathrm{Gr}}^\lambda_{\tilde F_x}(L)$ classifies $\mathcal{O}_x$-sublattices $L' \subset L$ such that $\dim(L/L') = d$. A point L' of this scheme lies in $\mathrm{Gr}^0_{\tilde F_x}$ if and only if $L' = t^{a_1}\mathcal{O}_x e_1 \oplus t^{a_2}\mathcal{O}_x e_2$ for some $a_1, a_2 \geq 0$ such that $d + m = a_1 + a_2$. So, the intersection in question identifies with the set of pairs $\{(a_1, a_2) \mid a_i \geq m,\ d + m = a_1 + a_2\}$. $\square$

Proof of Theorem 4 2) is easy and left to the reader.

1) We change the notation letting $\lambda = (0, -d) \in \Lambda^+_{\mathrm{GL}_2}$ for given $d \geq 0$. We will establish canonical isomorphisms

$$\mathrm{H}(\mathcal{A}_\lambda, \mathcal{W}_0) \widetilde{\to} \begin{cases} \mathcal{W}_d \otimes \tilde{E}_{\tilde{x}}^{\otimes -2d}, & \text{the nonsplit case, } \pi(\tilde{x}) = x \\ \\ \mathcal{W}_d \otimes \tilde{E}_{\tilde{x}_1}^{\otimes -d} \otimes \tilde{E}_{\tilde{x}_2}^{\otimes -d}, & \text{the split case, } \pi^{-1}(x) = \{x_1, x_2\} \end{cases}$$

Denote by K_m the $*$-restriction of $\mathrm{H}(\mathcal{A}_\lambda, \mathcal{W}_0)$ to $\mathcal{W}\mathrm{ald}_\pi^{x,m}$. Since $\mathcal{W}\mathrm{ald}_\pi^{x,0}$ is closed in $\mathcal{W}\mathrm{ald}_\pi^x$ and $\mathcal{W}_0$ is self-dual (up to replacing $\tilde{E}$ by $\tilde{E}^*$), our assertion is reduced to the following lemma. $\square$

Lemma 11. *We have $K_m = 0$ unless $m \leq d$. The complex K_m is placed in non positive (resp., strictly negative) perverse degrees for $m = d$ (resp., for $m < d$). We have canonically*

$$K_d \widetilde{\to} (\mathrm{pr}_{\mathcal{W}}^* A\tilde{E}) \otimes R \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{W}\mathrm{ald}_\pi^{x,d}},$$

where $\mathrm{pr}_{\mathcal{W}} : \mathcal{W}\mathrm{ald}_\pi^{x,d} \to \mathrm{Pic}\,\tilde{X}$ is the projection and

$$R \widetilde{\to} \begin{cases} \tilde{E}_{\tilde{x}}^{\otimes -2d}, & \textit{the nonsplit case, } \pi(\tilde{x}) = x \\ \\ \tilde{E}_{\tilde{x}_1}^{\otimes -d} \otimes \tilde{E}_{\tilde{x}_2}^{\otimes -d}, & \textit{the split case, } \pi^{-1}(x) = \{x_1, x_2\} \end{cases}$$

Proof Consider a point $\eta = (\mathcal{B}_{ex}, L \subset L_{ex} = \pi_* \mathcal{B}_{ex})$ of $\mathcal{W}\mathrm{ald}_\pi^{x,m}$, so $mx = \mathrm{div}(L_{ex}/L)$. Write ${}_x\overline{\mathcal{H}}^\lambda_{\mathrm{GL}_2}$ for the closed substack of ${}_x\mathcal{H}_{\mathrm{GL}_2}$ that under the projection $\mathfrak{q}_{\mathrm{GL}_2}$ identifies with

$$\mathrm{Bun}_2^x \times_{\mathrm{GL}_2(\mathcal{O}_x)} \overline{\mathrm{Gr}}^\lambda_{\mathrm{GL}_2} \to \mathrm{Bun}_2$$

Choose a trivialization of $\mathcal{B}_{ex}$ over $\mathrm{Spec}\,\tilde{\mathcal{O}}_x$. The fibre of $\mathfrak{p}_{\mathcal{W}} : \mathcal{W}\mathrm{ald}_\pi^x \times_{\mathrm{Bun}_2} {}_x\overline{\mathcal{H}}^\lambda_{\mathrm{GL}_2} \to \mathcal{W}\mathrm{ald}_\pi^x$ over η identifies with $\overline{\mathrm{Gr}}^{-w_0(\lambda)}_{\mathrm{GL}_2}(L)$, where we have set $-w_0(\lambda) = (d, 0)$. For the diagram

$$\mathcal{W}\mathrm{ald}_\pi^x \stackrel{\mathfrak{p}_{\mathcal{W}}}{\leftarrow} \mathcal{W}\mathrm{ald}_\pi^x \times_{\mathrm{Bun}_2} {}_x\overline{\mathcal{H}}^\lambda_{\mathrm{GL}_2} \stackrel{\mathfrak{q}_{\mathcal{W}}}{\to} \mathcal{W}\mathrm{ald}_\pi^x$$

we get $\mathrm{H}(\mathcal{A}_\lambda, \cdot) = (\mathfrak{p}_{\mathcal{W}})_! \mathfrak{q}_{\mathcal{W}}^*(\cdot)[d](\frac{d}{2})$. Only the stratum

$$\overline{\mathrm{Gr}}^{-w_0(\lambda)}_{\mathrm{GL}_2}(L) \cap \mathrm{Gr}^0_{\tilde{F}_x}$$

contributes to K_m. By Proposition 16, for $m = d$ this is a point whose image under $\mathfrak{q}_{\mathcal{W}}$ is

$$L' = \begin{cases} \pi_*(\mathcal{B}_{ex}(-2d\tilde{x})), & \text{the nonsplit case, } \pi(\tilde{x}) = x \\ \pi_*(\mathcal{B}_{ex}(-d\tilde{x}_1 - d\tilde{x}_2)), & \text{the split case, } \pi^{-1}(x) = \{\tilde{x}_1, \tilde{x}_2\} \end{cases}$$

Since $\dim \mathcal{W}\mathrm{ald}_\pi^{x,m} = m + \dim \mathrm{Pic}\,\tilde{X}$, our assertion follows from the automorphic property of $A\tilde{E}$. Namely, for the map $m_{\tilde{x}} : \mathrm{Pic}\,\tilde{X} \to \mathrm{Pic}\,\tilde{X}$ sending $\mathcal{B}$ to $\mathcal{B}(\tilde{x})$ we have canonically $m_{\tilde{x}}^* A\tilde{E} \widetilde{\to} A\tilde{E} \otimes \tilde{E}_{\tilde{x}}$. $\square$

Remarks . i) Our proof of Theorem 4 also shows the following. The stratum $\mathcal{W}\mathrm{ald}_\pi^{x,d}$ is dense in $\mathcal{W}\mathrm{ald}_\pi^{x,\le d}$. Besides, $\mathcal{W}_d[-\dim \mathcal{W}\mathrm{ald}_\pi^{x,d}]$ is a constructable sheaf on $\mathcal{W}\mathrm{ald}_\pi^{x}$ placed in usual cohomological degree zero. Its fibres over points of $\mathcal{W}\mathrm{ald}_\pi^{x,m}$ are 1-dimensional (resp., $d-m+1$-dimensional) in the non split (resp., split) case for $m\le d$.

ii) The category $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}_\pi^{x})$ is not semisimple. Indeed, for $\lambda=(0,-1)$ consider the finite map

$$\mathfrak{q}_{\mathcal{W}}:\mathcal{W}\mathrm{ald}_\pi^{x,0}\times_{\mathrm{Bun}_2}{}_x\overline{\mathcal{H}}^{\lambda}_{\mathrm{GL}_2}\to \mathcal{W}\mathrm{ald}_\pi^{x,\le 1}$$

It is an isomorphism over the open substack $\mathcal{W}\mathrm{ald}_\pi^{x,1}$. Since the open immersion $\mathcal{W}\mathrm{ald}_\pi^{x,1}\hookrightarrow \mathcal{W}\mathrm{ald}_\pi^{x,0}\times_{\mathrm{Bun}_2}{}_x\overline{\mathcal{H}}^{\lambda}_{\mathrm{GL}_2}$ is affine, the open immersion $\mathcal{W}\mathrm{ald}_\pi^{x,1}\hookrightarrow \mathcal{W}\mathrm{ald}_\pi^{x,\le 1}$ is also affine. Let $\mathcal{W}_{m,!}$ denote the !-extension of $\mathcal{W}_m\mid_{\mathcal{W}\mathrm{ald}_\pi^{x,m}}$ under (18). Then $\mathcal{W}_{1,!}\in \mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}_\pi^{x})$. So, if this category was semisimple, the exact sequence of perverse sheaves

$$0\to K\to \mathcal{W}_{1,!}\to \mathcal{W}_1\to 0$$

would split, which contradict the fact that the $*$-restriction $\mathcal{W}_1\mid_{\mathcal{W}\mathrm{ald}_\pi^{x,0}}$ is non zero.

8.6 Casselman-Shalika formula For $\lambda\in\Lambda^+_{\mathrm{GL}_2}$ write U^λ for the irreducible representation of the Langlands dual group $\check{\mathrm{G}}\mathrm{L}_2$ over $\bar{\mathbb{Q}}_\ell$. Let E be a $\check{\mathrm{G}}\mathrm{L}_2$-local system on X equiped with an isomorphism

$$U_E^{(1,1)}\widetilde{\to}\begin{cases}\tilde{E}^{\otimes 2}_{\tilde{x}}, & \text{the nonsplit case, } \pi(\tilde{x})=x\\ \tilde{E}_{\tilde{x}_1}\otimes\tilde{E}_{\tilde{x}_2}, & \text{the split case, } \pi^{-1}(x)=\{x_1,x_2\}\end{cases}$$

We associate to E the ind-object K_E of $\mathrm{P}^{\tilde{E}}(\mathcal{W}\mathrm{ald}_\pi^{x})$ given by

$$K_E=\mathop{\oplus}_{d\ge 0}\mathcal{W}_d\otimes U_E^{(0,-d)}$$

For a representation U of $\check{\mathrm{G}}\mathrm{L}_2$ write $\mathcal{A}_U$ for the object of $\mathrm{Sph}(\mathrm{Gr}_{\mathrm{GL}_2})$ corresponding to V via the Satake equivalence $\mathrm{Rep}(\check{\mathrm{G}}\mathrm{L}_2)\widetilde{\to}\mathrm{Sph}(\mathrm{Gr}_{\mathrm{GL}_2})$.

One formally derives from Theorem 4 the following.

Corolary 3. *For any $U\in\mathrm{Rep}(\check{\mathrm{G}}\mathrm{L}_2)$ there is an isomorphism $\alpha_U:\mathrm{H}(\mathcal{A}_U,K_E)\widetilde{\to}K_E\otimes U_E$. For $U,U'\in\mathrm{Rep}(\check{\mathrm{G}}\mathrm{L}_2)$ the diagram commutes*

$$\begin{array}{ccc}\mathrm{H}(\mathcal{A}_{U'},\mathrm{H}(\mathcal{A}_U,K_E)) & \stackrel{\alpha_U}{\to} & \mathrm{H}(\mathcal{A}_{U'},K_E\otimes U_E)\\ \downarrow\gamma & & \downarrow\alpha_{U'}\otimes\mathrm{id}\\ \mathrm{H}(\mathcal{A}_{U\otimes U'},K_E) & \stackrel{\alpha_{U\otimes U'}}{\to} & K_E\otimes(U\otimes U')_E,\end{array}$$

where γ is the isomorphism (19).

Remark 8. One may view $\mathrm{Gr}_{\tilde{F}_x}$ as the ind-scheme classifying a rank 2 vector bundle L on X together with an isomorphism $L \widetilde{\rightarrow} \pi_* \mathcal{O}_{\tilde{X}} \mid_{X-x}$. This yields a natural map $\mathrm{Gr}_{\tilde{F}_x} \to \mathcal{W}\mathrm{ald}^x_\pi$.

The results of Sect. 8 hold also in the case of a finite base field $k = \mathbb{F}_q$. In this case we have the Waldpurger module WA_χ introduced in 1.4. For $d \geq 0$ consider the function trace of Frobenius of $\mathcal{W}_d$ on $\mathcal{W}\mathrm{ald}^x_\pi(k)$, let W_d be its restriction to $\mathrm{Gr}_{\tilde{F}_x}$. Then $\{W_d, d \geq 0\}$ is a base of the vector space WA_χ.

The space WA_χ also has the base (indexed by $d \geq 0$) consisting of functions supported over the $\tilde{F}^*_x$-orbit corresponding to d. The Casselman-Shalika formula in this base is given by ([3], Theorem 1.1), it involves some nontrivial denominators. This corresponds to the fact that our ind-object K_E is not locally finite on $\mathcal{W}\mathrm{ald}^x_\pi$.

Appendix A. Fourier transforms

For the convenience of the reader, we collect some well-known observations about equivariant categories and Fourier transforms that we need. The proofs are omitted.

A.1 Let S be a scheme of finite type and $\mathrm{pr} : G \to S$ be a groupoid. Assume that pr is of finite type, with contractible fibres and smooth of relative dimension k. Assume also that $\mathrm{act} : G \to S$ is smooth of relative dimension k. Let $\mathcal{L}$ be a local system on G whose restriction to the unit section $S \to G$ is trivialized.

By ([5], Lemma 4.8), we have the Serre subcategory $\mathrm{P}^W(S) \subset \mathrm{P}(S)$ of perverse sheaves $K \in \mathrm{P}(S)$ such that there exists an isomorphism $\mathrm{act}^* K \otimes \mathcal{L} \widetilde{\rightarrow} \mathrm{pr}^* K$ whose restriction to the unit section is the identity. Let $\mathrm{D}^W(S) \subset \mathrm{D}(S)$ denote the full triangulated subcategory generated by $\mathrm{P}^W(S)$.

We write $\mathrm{D}^W_{\mathcal{L}}(S)$ if we need to express the dependence on $\mathcal{L}$. For $K \in \mathrm{D}(S)$ we have $K \in \mathrm{D}^W_{\mathcal{L}}(S)$ if and only if $\mathbb{D}(K) \in \mathrm{D}^W_{\mathcal{L}^*}(S)$.

Let $\beta : S' \to S$ be an S-scheme of finite type. The groupoid G "lifts" to S' if we have two cartesian squares

$$\begin{array}{ccc} G & \stackrel{\mathrm{pr}}{\to} & S \\ \uparrow \beta' & & \uparrow \beta \\ G' & \stackrel{\mathrm{pr}'}{\to} & S' \end{array}$$

and

$$\begin{array}{ccc} G & \stackrel{\mathrm{act}}{\to} & S \\ \uparrow \beta' & & \uparrow \beta \\ G' & \stackrel{\mathrm{act}'}{\to} & S' \end{array}$$

that make G' a groupoid over S'.

For the local system $\beta'^* \mathcal{L}$ we get the category $\mathrm{D}^W(S')$. The functors $\beta_!$ and β_* send $\mathrm{D}^W(S')$ to $\mathrm{D}^W(S)$. The functors β^* and $\beta^!$ send $\mathrm{D}^W(S)$ to $\mathrm{D}^W(S')$.

A.2 Let $Y \to Z$ be a morphism of schemes of finite type and $E \to Z$ be a vector bundle over Z. Assume that E acts on Y over Z, and $\mathrm{act} : E \times_Z Y \to Y$ is smooth of relative dimension $\mathrm{rk}\, E$.

We have a natural pairing $\chi : E^* \times_Z E \times_Z Y \to \mathbb{A}^1$. For the local system $\mathcal{L} = \chi^*\mathcal{L}_\psi$ we get the category $\mathrm{D}^W(E^* \times_Z Y)$ as in A.1.

Let $F : \mathrm{D}(Y) \to \mathrm{D}^W(E^* \times_Z Y)$ be the functor

$$F(K) = \mathrm{Four}(\mathrm{act}^* K)[\mathrm{rk}\, E](\frac{\mathrm{rk}\, E}{2})$$

Then F is an equivalence of triangulated categories, t-exact and commutes with Verdier duality (up to replacing ψ by ψ^{-1}). The quasi-inverse functor is given by $K \mapsto \mathrm{pr}_!(K)$, where $\mathrm{pr} : E^* \times_Z Y \to Y$ is the projection.

Moreover, for any $K \in \mathrm{D}^W(E^* \times_Z Y)$ the natural map $\mathrm{pr}_!(K) \widetilde{\to} \mathrm{pr}_*(K)$ is an isomorphism.

A.3 Suppose we are in the situation of A.2. Assume in addition that $p : E' \to E$ is a morphism of vector bundles over Z. Then E' also acts on Y over Z (via E), and we have the functor $F' : \mathrm{D}(Y) \to \mathrm{D}^W(E'^* \times_Z Y)$ defined as in A.2.

Then we have an isomorphism of functors $F' \widetilde{\to} (\check{p}\times\mathrm{id})_! \circ F$, where $\check{p}\times\mathrm{id} : E^*\times_Z Y \to E'^*\times_Z Y$ is the dual map (cf. [5], 5.16).

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References

[1] A. Beilinson, V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves, preprint

[2] A. Braverman, D. Gaitsgory, Geometric Eisenstein series, Invent. Math. 150 (2002), no. 2, 287–384.

[3] D. Bump, S. Friedberg, M. Furusawa, Explicit formulas for the Waldspurger and Bessel models, math.RT/9410202

[4] E. Frenkel, D. Gaitsgory, K. Vilonen, Whittaker patterns in the geometry of moduli spaces of bundles on curves, Ann. of Math. (2) 153 (2001), no. 3, 699–748.

[5] D. Gaitsgory, On a vanishing conjecture appearing in the geometric Langlands correspondence, math.AG/0204081

[6] S. Lysenko, On automorphic sheaves on Bun_G, math.RT/0211067

[7] I. Mirkovic, K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, math.RT/0401222

[8] C. Moeglin, M.-F. Vigneras, J.L. Waldspurger, Correspondence de Howe sur un corps p-adique, Lecture Notes in Math. 1291 (1987)

[9] I.I.Piatetski-Shapiro, On the Saito-Kurokawa lifting, Invent. Math. 71 (1983), 309-338

[10] Rallis, S. On the Howe duality conjecture. Compositio Math. 51 (1984), no. 3, 333–399.

[11] J.-L. Verdier, Des catégories dérivées des catégories abéliennes, Asterisque 239 (1996), Société Math. de France

[12] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphe en leur centre de symétrie, Compositio Math. 54 (1985), 173-242